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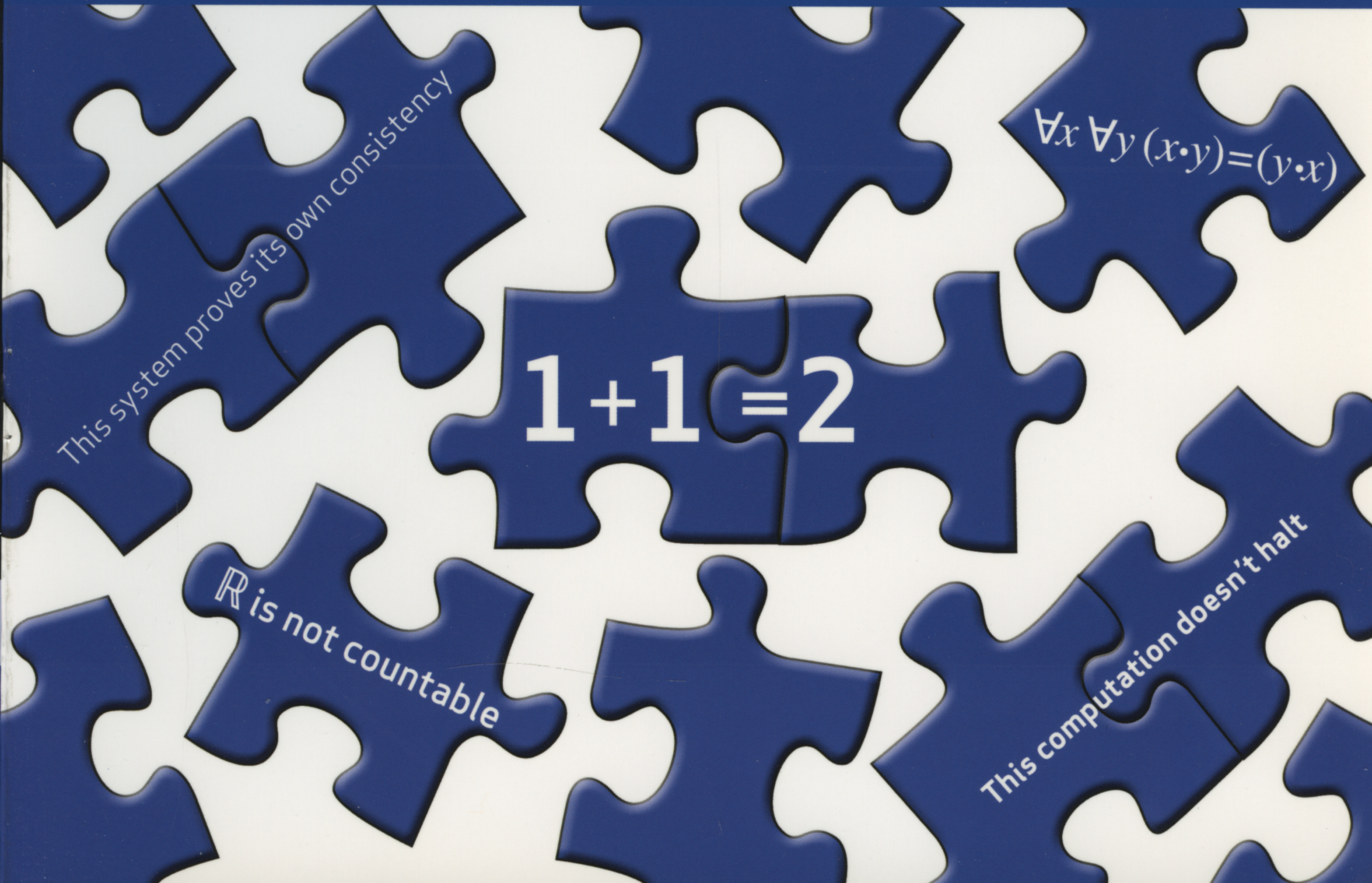
M381 Number Theory and
Mathematical Logic

Mathematical Logic **Unit 5**

Formal Proof


This system proves its own consistency
∀x ∀y (x·y)=(y·x)
1+1=2
ℝ is not countable
This computation doesn't halt

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Prepared by the Course Team

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$\forall x \forall y\, (x \cdot y) = (y \cdot x)$
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This computation doesn't halt

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CONTENTS

Introduction **4**

1 First Steps **4**

  1.1 Mathematical proof 4

  1.2 Three rules of proof 8

2 Quantifier Logic: The Universal Quantifier **16**

  2.1 Free and bound variables 16

  2.2 The Universal Quantifier Elimination Rule 20

  2.3 The Universal Quantifier Introduction Rule 24

3 Quantifier Logic: The Existential Quantifier **28**

  3.1 The Existential Quantifier Introduction Rule 28

  3.2 The Existential Hypothesis Rule 30

Summary **35**

Objectives **36**

Additional Exercises **37**

Solutions to the Problems **40**

Solutions to Additional Exercises **45**

Index **50**

INTRODUCTION

In *Unit 4* we described the formal language we shall be using in the remainder of the course. We also introduced the idea of an interpretation. Now we introduce the notion of a *formal proof* of a formula of our formal language. We want to specify when we can derive or prove a given formula from other formulas used as assumptions.

We shall often call a proof within our formal system a *derivation* and use the word *derive* to describe the process of giving such a proof.

A formal proof of a formula ψ from formulas $\phi_1, \phi_2, \ldots, \phi_k$ used as assumptions is a list of formulas constructed as follows. We begin by listing the formulas $\phi_1, \phi_2 \ldots, \phi_k$. We describe nine logical *rules of proof* that tell us which other formulas we can add to the list. For example, one rule, called the *Tautology Rule*, tells us that if a formula θ is a tautological consequence of formulas already in the list, then θ may be added to the list. If we can produce a list of formulas constructed from the assumptions using the rules and which ends with the formula ψ, then we have derived ψ from $\phi_1, \phi_2 \ldots, \phi_k$.

An obvious requirement for formal proofs is that they should be logically valid, in the sense made precise in Subsection 3.3 of *Unit 4*: if we derive a formula ψ from formulas $\phi_1, \phi_2 \ldots, \phi_k$, then ψ should be a logical consequence of $\phi_1, \phi_2 \ldots, \phi_k$; that is, we require that ψ should be true in every interpretation in which all of the formulas $\phi_1, \phi_2 \ldots, \phi_k$ are true. A second requirement is that there should be an algorithm for deciding whether a given list of formulas is a formal proof, so that this can be checked by a machine.

In this unit, we introduce seven of the rules of proof. We shall complete the list of the logical rules of our formal system and start looking at how to use it to prove statements about number theory in *Unit 6*.

1 FIRST STEPS

We shall begin this section by examining an informal proof from everyday mathematics to see the way in which it might be formalized. Then we shall introduce our first three rules of proof.

1.1 Mathematical proof

In the Introduction to *Unit 4* we remarked that, whereas in everyday mathematics we do not state explicitly which logical principles we are using, in a formal system the rules that can be used in carrying out deductions are stated explicitly. Before we describe, in this unit and the next, the rules that we are going to allow, we need to think about the nature of an informal mathematical proof. Our intention is that our precisely defined notion of *formal proof* should correspond as closely as possible to the informal notion used by mathematicians.

Since the informal notion is not absolutely precise, we cannot hope to give an exact definition, but we feel that there would not be too much disagreement with the following description.

> A mathematical proof is a finite sequence of mathematical assertions which forms a valid and convincing argument for the desired conclusion from stated assumptions.

We shall sometimes use 'valid' as shorthand for 'logically valid'.

In this description 'finite sequence' means that the finitely many assertions are in some definite order, with a beginning and an end. 'Valid' means that the conclusion does really follow from the stated assumptions, that is, the conclusion is a logical consequence of the assumptions. It is because proofs are valid arguments that, when we have proved something, we know it is true, given that the assumptions are true.

The validity of an argument is a matter of logical fact; in contrast, 'convincing' is a psychological notion. From our point of view this is unsatisfactory. We want it to be a matter of objective fact that a sequence of assertions counts as a formal proof. Each step should be checkable by another mathematician working in a mechanical way. Indeed, we require that there is an algorithm to decide whether or not a sequence of formulas satisfies the requirements of being a formal proof, so that it can be checked by a machine.

As we want formal proofs to correspond to the proofs we use in everyday mathematics, let us look at such a proof to see how it can be reorganized to bring out its logical structure. The example we have chosen comes from the beginnings of number theory.

Theorem

If the square of a natural number is even, then the number itself is even.

Proof

Assume n^2 is even. Next assume that n is odd. Then, by definition, there is a natural number k such that $n = 2k + 1$. Hence $n^2 = (2k + 1)^2$. But $(2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$, so $n^2 = 2(2k^2 + 2k) + 1$. This shows that n^2 is odd, so is not even, contradicting our initial assumption. Hence n cannot be odd, so n is even. ■

The above is a perfectly good and correct everyday proof. It is not, however, in the best form for a machine to check that it is indeed correct. Apart from not being written in a formal language, the proof leaves unsaid much of the justification of the steps that it uses. For instance, when we write '$n^2 = (2k + 1)^2$. But $(2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$, so $n^2 = 2(2k^2 + 2k) + 1$', we are assuming familiarity with the rule that allows us to deduce that $a = c$ from the equations $a = b$ and $b = c$ between natural numbers a, b and c. Of course, in everyday mathematics it is perfectly reasonable to assume that a reader will take this for granted. A machine, however, cannot take it for granted; a machine needs the rule and the use of the rule to be spelled out.

Thus, as a first step towards our description of a formal proof, let us rewrite the proof above in a different format. First, we shall use a separate line for each assertion made in the argument, for convenience changing the words in places. Second, we shall give each line a number corresponding to the position of the line in the sequence of assertions. Next, to the right of each assertion we shall write down the justification for that assertion and, as this justification will often be in terms of earlier assertions, the line numbering is particularly useful here. Our rewritten proof is as follows.

This is, of course, quite common practice in mathematics, for instance when one wishes to refer to important equations.

Second version of the proof

Line number	*Assertion*	*Justification*
(1)	n^2 is even	Assumption
(2)	n is odd	Assumption
(3)	For some number k, $n = 2k + 1$	From 2 using the definition of 'odd'
(4)	$n^2 = (2k + 1)^2$	From 3
(5)	$(2k + 1)^2 = 4k^2 + 4k + 1$	Algebraic manipulations
(6)	$4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$	
(7)	$(2k + 1)^2 = 2(2k^2 + 2k) + 1$	From 5 and 6
(8)	$n^2 = 2(2k^2 + 2k) + 1$	From 4 and 7
(9)	For some number l, $n^2 = 2l + 1$	From 8 putting $l = 2k^2 + 2k$
(10)	n^2 is odd	From 9, by the definition of 'odd'
(11)	n^2 is odd and n^2 is even	From 10 and 1
(12)	n is not odd	From 11, using 'proof by contradiction'
(13)	n is even	From 12
(14)	n^2 is even implies n is even	13 has been deduced from 1

The way that this layout separates the assertions from their justifications, that is, the rules used to arrive at the assertions from previous ones, is a feature that we shall build into our idea of a formal proof. A formal proof will consist of a sequence of assertions, each of which is accompanied by a reference to the rule by which it was derived from previous assertions.

It seems that in our informal proof we are using a large number of different rules for deriving assertions. We shall see in this unit and *Unit 6* that, for formal proofs, we can get by with just nine rules. Before we start looking at these rules there is one simple, but important, refinement that we make to the format of our proof.

Most proofs, formal and informal, contain assertions that are, in fact, assumptions, such as lines 1 and 2 in the proof above. Various conclusions are drawn from these assumptions. For example, at line 10 we have concluded from the assumption that 'n is odd' that 'n^2 is odd', which has led us in line 11 to conclude from the assumption that 'n^2 is even' that 'n^2 is odd and n^2 is even', giving us a contradiction. We concluded that it cannot be the case that both initial assumptions are simultaneously true. It was relatively easy to see in this proof that both initial assumptions were used in deriving the contradictory assertion at line 11, which in turn enabled us to conclude that both assumptions cannot hold simultaneously. In other proofs it may not be so easy, and yet it is important to know which assumptions are implicit in any given assertion. To this end, it is going to be very helpful to keep a record, on each line of the proof, of which assumptions have been used in deriving the assertion on that line.

Keeping track of which assumptions have been used to derive the assertion on a particular line is not too difficult. For example, if in the above proof we look at the justification for line 8, we see this line has been deduced from lines 4 and 7. Now line 4 follows from line 3 which, in turn, has been deduced from the assumption on line 2, which depends on nothing earlier in the proof. Also line 7 follows from lines 5 and 6, which are justified by simple algebraic properties of the natural numbers (such as the distributive law, the commutative law and so on). Let us write AP for the algebraic properties used here; they are not really our main concern at this stage, although in a deeper analysis of this proof one would have to investigate exactly which of these properties are being used. Thus we see that the assertion on line 8 depends ultimately on the assumption on line 2 and AP. We usually express this by saying that the *assumptions in force* on line 8 are line 2 and AP, or that line 8 *depends* on the assumptions 2 and AP. We are going to indicate this by writing '2, AP' to the left of the line number. If we indicate the assumptions in force on each line in this way, we obtain the following version of our proof.

Third version of the proof

Assumptions	*Line number*	*Assertion*	*Justification*
1	(1)	n^2 is even	Assumption
2	(2)	n is odd	Assumption
2	(3)	For some number k, $n = 2k + 1$	From 2 using the definition of 'odd'
2	(4)	$n^2 = (2k+1)^2$	From 3
AP	(5)	$(2k+1)^2 = 4k^2 + 4k + 1$	Algebraic manipulations
AP	(6)	$4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$	
AP	(7)	$(2k+1)^2 = 2(2k^2 + 2k) + 1$	From 5 and 6
2, AP	(8)	$n^2 = 2(2k^2 + 2k) + 1$	From 4 and 7
2, AP	(9)	For some number l, $n^2 = 2l + 1$	From 8 putting $l = 2k^2 + 2k$
2, AP	(10)	n^2 is odd	From 9, by the definition of 'odd'
1, 2, AP	(11)	n^2 is odd and n^2 is even	From 10 and 1
1, AP	(12)	n is not odd	From 11 using 'proof by contradiction'
1, AP	(13)	n is even	From 12
AP	(14)	n^2 is even implies n is even	13 has been deduced from 1

We have written the number 1 at the extreme left of line 1 because the assertion 'n^2 is even' is an assumption and so naturally the assumption in force on line 1 is just the assertion that occurs there. Similarly the assumption in force on line 2 is just the assertion on this line. To find the assumptions in force on line 3, we look at the justification for this line. We see that line 3 follows from line 2 and so depends on the same assumptions as line 2. So we write the number 2 to the left of line 3. Similarly line 4 follows from line 3, and so depends on the same assumptions that line 3 depends on. Since the only assumption in force on line 3 is the one on line 2, we write the number 2 to the left of line 4.

We carry on in this way, keeping track of the assumptions in force on each line. We leave it to you to verify that the details written to the left of lines 5 to 10 correspond to the assumptions in force on those lines.

When it comes to line 11, we see that this line follows from lines 10 and 1. So the assumptions in force on this line are all those in force on line 10, namely 2, AP, *together* with the assumption in force on line 1, namely 1. Thus we have written 1, 2, AP to the left of the line 11 to indicate that these are the assumptions in force on that line. Now, what have we proved at this stage of the argument? Our analysis reveals, on line 11, that the assumptions 1, 2, AP lead to a contradictory statement, namely 'n^2 is odd and n^2 is even'. Hence not all these assumptions can be true simultaneously. So if the assertion on line 1 and AP are assumed to be true, then it follows that the assertion on line 2 must be false, so that n cannot be odd. This is precisely what we have indicated on line 12, where we specify that the assertion 'n is not odd' can be deduced from the assumptions 1, AP.

The proof is essentially complete on line 13 since this line states that, assuming AP, the assertion 'n is even' follows from the assertion on line 1, namely that 'n^2 is even'. However, it is neater to have the actual statement of the theorem we are proving as the last line of the proof. So we incorporate assumption 1 into the assertion on line 14, leaving AP as the only assumptions in force on this line.

The last three lines in the third version of the proof illustrate the advantages of keeping track of the assumptions in force on each line. A casual glance at line 13 in the second version of the proof might suggest that we have proved the assertion 'n is even' without any restriction. This would be strange as natural numbers are not all even. However, the analysis in the third version brings out the fact that we have only proved 'n is even' *assuming* that 'n^2 is even', and this is made perfectly clear in line 13 of the third version. Also, as AP remains to the left of line 14, we have revealed that, not surprisingly, some algebraic properties of the natural numbers have been used in the proof.

Note that some steps in the argument add to the number of assumptions in force, for example those on lines 8 and 11, whereas others reduce the number of assumptions, for example those on lines 12 and 14.

The third version of our proof is still essentially informal. Nevertheless our notion of formal proof will copy its layout, so that a *formal proof* will consist of a finite table with columns for:

Assumptions *Line number* *Assertion* *Justification*

We shall not include the column headings when writing down a formal proof.

An *assertion* in a formal proof will be simply a formula of the formal language, as defined in *Unit 4*. The *justification* will consist of a reference to the use of one of the nine *rules of proof* that we shall define and, in most cases, to the earlier line or lines of the proof to which the rule has been applied. The rules will specify which formulas may be derived, and hence written in the assertion column, and which assumptions are in force on the new line. The last line of a formal proof will contain the assertion, or result, that has been formally proved from the assumptions in force on that line.

In the rest of this unit and in *Unit 6* we shall look at the rules that can be used in a formal proof, and at how we make use of them.

1.2 Three rules of proof

When we come to choose which rules of proof to allow, we must bear in mind that it is an absolute requirement that formal proofs should be logically valid and that they can be checked by a machine. Before we discuss logical validity and machine checkability, however, we need to consider a practical matter.

Practical considerations pull us in two directions. When it comes to *giving* formal proofs, we would like our formal system to be very powerful with lots of rules, so that formal proofs are easy to construct and so that a formal proof exists whenever we can reasonably expect this. From this point of view the only constraints are those of logical validity and machine checkability mentioned above. However, when it comes to *studying* a formal system, as an object of mathematical enquiry, it is desirable that the formal system should be as simple as possible, with a small number of rules. In this course we have chosen to use a formal system with rather a small number of rules, but which is sufficiently powerful for formal proofs of formulas to exist whenever this is not ruled out by the requirement of logical validity.

We encountered a similar dilemma when choosing the basic symbols of our formal system in *Unit 4* (Subsection 1.1).

We now need to explain exactly what the requirement of *logical validity* means. As we have noted, a formal proof establishes the claim that a certain formula can be deduced from a set of formulas used as assumptions. The requirement of logical validity is that, whenever we have a formal proof of a formula ψ from a set of assumptions $\phi_1, \phi_2, \ldots, \phi_k$, then ψ must be a logical consequence of these assumptions. You should recall that this means that ψ is true in every interpretation in which $\phi_1, \phi_2, \ldots, \phi_k$ are true. We shall ensure that formal proofs are logically valid by ensuring that each individual rule we introduce preserves logical validity, so that each step within a proof preserves logical validity.

See *Unit 4*, Definition 3.4.

We now come to our second requirement, that of *machine checkability.* We incorporate this requirement into our formal system by designing each of the rules of proof in such a way that there is an algorithm that can check whether a purported use of the rule is a correct application of it.

We are now ready to introduce our first rule of proof. It corresponds to the one used on lines 1 and 2 of the informal proof given in Subsection 1.1, that is the rule which corresponds to making an assumption. For this reason it is called the Assumption Rule.

Definition 1.1 Assumption Rule (Ass)

Any formula may be introduced on a line of a formal proof. The only assumption in force on this line is the formula itself.

We write Ass in the *Justification* column of a formal proof to indicate that the Assumption Rule has been used. Thus a line on which this rule is used will have the following form.

Assumptions	*Line number*	*Assertion*	*Justification*
k	(k)	ϕ	Ass

Note that *the assumption number is the same as the line number.* This is because, as specified by the rule, the only assumption in force on a line where the Assumption Rule is used is the formula introduced as an assumption on that line.

You may think that the Assumption Rule is rather trivial. It is a very simple rule, but it should not be undervalued. We used it twice in our informal proof. We shall subsequently see that, without this rule, most formal proofs would not get off the ground. In most cases the first line of a formal proof will consist of an application of this rule.

It is easily seen that the Assumption Rule meets the requirements of logical validity and machine checkability. When we use the rule, a formula ϕ is derived from the assumption ϕ. It is trivially true that a formula ϕ is a logical consequence of ϕ. So the rule is logically valid. It is also clear that there is an algorithm to check whether a line in a formal proof has the correct form for a use of this rule, as set out above.

The next rule that we introduce is more powerful. We first specify the rule, then we give an example of its use, after which we explain why it satisfies our two requirements, namely logical validity and machine checkability.

You will need to recall the definition of tautological consequence, namely that the formula ψ is a tautological consequence of the formulas $\phi_1, \phi_2, \ldots, \phi_k$ if the formula

$$((\cdots((\phi_1 \,\&\, \phi_2) \,\&\, \phi_3) \cdots \&\, \phi_k) \to \psi)$$

is a tautology.

See *Unit 4*, Definition 3.5.

Definition 1.2 *Tautology Rule (Taut)*

If the formulas $\phi_1, \phi_2, \ldots, \phi_k$ occur on certain lines of a formal proof and the formula ψ is a tautological consequence of $\phi_1, \phi_2, \ldots, \phi_k$, then on any subsequent line we may introduce the formula ψ, which will depend on all the assumptions in force on the lines on which the formulas $\phi_1, \phi_2, \ldots, \phi_k$ occur.

When we use the Tautology Rule we indicate this in the *Justification* column by writing Taut followed by the numbers of the lines on which the formulas $\phi_1, \phi_2, \ldots, \phi_k$ occur.

Here is an example to illustrate the use of this rule.

Example 1.1

1	(1)	$(\exists x\,(x+y)=\mathbf{0} \to y=\mathbf{0'})$	Ass
2	(2)	$(\exists x\,(x+y)=\mathbf{0} \vee \forall z\, y=z)$	Ass
1, 2	(3)	$(y=\mathbf{0'} \vee \forall z\, y=z)$	Taut, 1, 2

The annotation on the right of line 3 indicates that we have derived this line using the Tautology Rule from the formulas on lines 1 and 2. Thus the assumptions in force on line 3 are all those in force on lines 1 and 2, namely 1 and 2. This is indicated by the annotation 1, 2 on the left of line 3.

For this to be a correct use of the Tautology Rule, we need the formula

$$(((\exists x\,(x+y)=\mathbf{0} \to y=\mathbf{0'})\ \&\ (\exists x\,(x+y)=\mathbf{0} \vee \forall z\, y=z)) \to (y=\mathbf{0'} \vee \forall z\, y=z))$$

to be a tautology. We have already seen, in *Unit 4*, Problem 3.2(b), that this formula is a tautology. So the table above is a correct formal proof. It is a formal proof of the formula on line 3 from the assumptions in force on this line, namely the formulas on lines 1 and 2. ♦

Now we check that the Tautology Rule meets our requirement of logical validity. We need to show that if a formal proof is valid up to the point at which we make use of this rule, then it remains valid to derive the formula ψ from the formulas $\phi_1, \phi_2, \ldots, \phi_k$ which occur on specified earlier lines of the proof, on which the sets of assumptions in force are $\Delta_1, \Delta_2, \ldots, \Delta_k$ respectively. The supposition that the proof is valid at this point means that, for $1 \le i \le k$, the formula ϕ_i is a logical consequence of the set of formulas Δ_i.

Δ is the Greek letter 'capital delta'.

When we apply the Tautology Rule to derive the formula ψ, the set of assumptions in force will be all those in the sets $\Delta_1, \Delta_2, \ldots, \Delta_k$. That is, the set of assumptions is Δ where

$$\Delta = \Delta_1 \cup \Delta_2 \cup \cdots \cup \Delta_k.$$

We show that the formal proof is still valid by showing that, in this situation, ψ is a logical consequence of the set of formulas Δ. We thus need to show that if we have an interpretation in which all the formulas in Δ are true, then ψ is also true in this interpretation.

So suppose that we have an interpretation in which all the formulas in Δ are true. Then, for $1 \le i \le k$, all the formulas in Δ_i are true, and hence, as ϕ_i is a logical consequence of Δ_i, ϕ_i is also true in this interpretation. Thus each of the formulas $\phi_1, \phi_2, \ldots, \phi_k$ is true. As we have used the Tautology Rule to derive ψ, ψ is a tautological consequence of $\phi_1, \phi_2, \ldots, \phi_k$. Hence, by Theorem 3.1 of *Unit 4*, it is a logical consequence of these formulas. It follows that ψ is also true in the given interpretation.

This completes the proof that ψ is a logical consequence of Δ. In this way we have shown that the Tautology Rule is logically valid.

A use of the Tautology Rule is valid if and only if the relevant formula $((\cdots((\phi_1 \mathbin{\&} \phi_2) \mathbin{\&} \phi_3)\cdots \mathbin{\&} \phi_k) \rightarrow \psi)$ is a tautology. We saw in *Unit 4* that there is an algorithmic method based on truth tables for determining whether or not a formula is a tautology. Thus the Tautology Rule also meets our requirement of machine checkability.

The Tautology Rule is quite powerful. It encompasses in a single rule almost all deductions whose logical validity arises from the meanings of the connectives. We made use of an informal version of the Tautology Rule in the informal proof which we gave in Subsection 1.1. Consider the step made at line 11. If we use ϕ_1 to represent 'n^2 is odd' and ϕ_2 to represent 'n^2 is even' then we see that this step amounts to deducing the formula $(\phi_1 \mathbin{\&} \phi_2)$ from the formulas ϕ_1, ϕ_2. Since, as you can easily check, the formula $((\phi_1 \mathbin{\&} \phi_2) \rightarrow (\phi_1 \mathbin{\&} \phi_2))$ is a tautology, this step corresponds to a use of the Tautology Rule.

In many standard treatments of logic the Tautology Rule is replaced by a number of different rules dealing separately with each connective. Both approaches have their advantages and disadvantages. One major advantage of using the Tautology Rule is that it generally makes formal proofs shorter.

The next rule that we introduce also corresponds to a step in the informal proof we have just been discussing. This is the step at line 14, where we incorporated an assumption into the assertion by using the word 'implies'. Since it is the symbol → that corresponds to 'implies', we formulate our third rule, called the Conditional Proof Rule, as follows.

Definition 1.3 Conditional Proof Rule (CP)

If the formula ψ occurs on a line of a formal proof and the formula ϕ occurs among the assumptions in force on that line, then on any subsequent line we may introduce the formula $(\phi \rightarrow \psi)$, which will depend on all the assumptions other than ϕ which are in force on the line on which the formula ψ occurs.

When we use this rule we indicate this in the *Justification* column by writing CP followed by the number of the line on which the formula ψ occurs.

Here is an example to illustrate the use of this rule.

Example 1.2

We give a formal proof of the formula

$$(x = y \rightarrow (x = y \lor y = \mathbf{0}))$$

depending on no assumptions at all.

1	(1)	$x = y$	Ass
1	(2)	$(x = y \lor y = \mathbf{0})$	Taut, 1
	(3)	$(x = y \rightarrow (x = y \lor y = \mathbf{0}))$	CP, 2

The use of the Tautology Rule on line 2 is justified as the formula $(x = y \rightarrow (x = y \lor y = \mathbf{0}))$ is a tautology, as you can easily check.

The use of the Conditional Proof Rule on line 3 is justified as follows. On line 2 we have the formula $(x = y \lor y = \mathbf{0})$ and the formula $x = y$ is in force as an assumption on this line. The Conditional Proof rule tells us that we can introduce the formula $(x = y \rightarrow (x = y \lor y = \mathbf{0}))$ and that this depends on all the assumptions on which $(x = y \lor y = \mathbf{0})$ depends other than the formula $x = y$. But there are no assumptions on which $(x = y \lor y = \mathbf{0})$ depends, on line 2, other than $x = y$. So the formula on line 3 does not depend on any assumptions at all. We indicate this by not writing anything in the *Assumptions* column on the left of line 3. On the right of this line we write CP, 2 in the *Justification* column to indicate that line 3 has been obtained by applying the Conditional Proof Rule to line 2. ◆

As with the other rules, we need to check the logical validity of the Conditional Proof Rule. Suppose that we have a formal proof which is valid to the point at which we make use of this rule, and that we apply this rule to derive the formula $(\phi \to \psi)$ from an earlier line on which ψ has been derived from a set of assumptions, $\phi, \phi_1, \phi_2, \ldots, \phi_k$ say, which includes ϕ. Our supposition that the proof so far is valid means that ψ is a logical consequence of $\phi, \phi_1, \phi_2, \ldots, \phi_k$. When we use the Conditional Proof Rule we derive the formula $(\phi \to \psi)$ from the set of assumptions $\phi_1, \phi_2, \ldots, \phi_k$. Thus to show that the proof is still valid we need to show that $(\phi \to \psi)$ is a logical consequence of $\phi_1, \phi_2, \ldots, \phi_k$.

To this end we consider an interpretation in which all the formulas $\phi_1, \phi_2, \ldots, \phi_k$ are true. In this interpretation the formula ϕ is either true or false. We show that in either case the formula $(\phi \to \psi)$ is true. If ϕ is false in this interpretation, then it follows immediately from the truth table for → that $(\phi \to \psi)$ is true. If ϕ is true then in this interpretation all the formulas $\phi, \phi_1, \phi_2, \ldots, \phi_k$ are true. Then it follows immediately from our supposition that ψ is a logical consequence of $\phi, \phi_1, \phi_2, \ldots, \phi_k$ that ψ is true. Hence, using the truth table for →, the formula $(\phi \to \psi)$ is true. Thus in either case $(\phi \to \psi)$ is true. It follows that this formula is a logical consequence of $\phi_1, \phi_2, \ldots, \phi_k$, and hence that the Conditional Proof Rule is logically valid.

It is equally easy to see that the Conditional Proof Rule meets our machine checkability requirement. Suppose that this rule has been used on a certain line. To check that a correct use of the rule has been made, two things need to be checked. First, the formula that has been derived must have the form $(\phi \to \psi)$ where ψ is the formula which occurs on the earlier line referred to in the justification and where ϕ is one of the assumptions in force on this line. Second, the assumptions which are listed as being in force on the line on which $(\phi \to \psi)$ occurs must be precisely the assumptions on the earlier line other than ϕ. It should be evident that both checks can be carried out algorithmically.

You should note that whenever we use the Conditional Proof Rule one assumption formula is removed. Since in most formal proofs the Assumption Rule will be used to introduce assumptions, it is useful also to have a rule which allows assumptions to be eliminated.

Schematic proofs

Look back at the formal proofs given in Examples 1.1 and 1.2. The validity of each proof depends on the meanings given to the connectives used in the formulas in these proofs. These connectives are used to build up formulas from subformulas which do not contain any connectives. In order to give explicit examples we had to specify the details of these subformulas, but the validity of these proofs does not depend on these details in any way.

For example, in Example 1.2 we noted that the formula $(x = y \to (x = y \vee y = \mathbf{0}))$ is a tautology. We can show this by letting θ and χ represent $x = y$ and $y = \mathbf{0}$ respectively, so that the formula becomes $(\theta \to (\theta \vee \chi))$, and then drawing up a truth table for this formula. However, the formula $(\theta \to (\theta \vee \chi))$ is a tautology *whatever formulas are substituted for θ and χ*. This means that the following is a valid formal proof whatever formulas θ and χ happen to be.

1	(1)	θ	Ass
1	(2)	$(\theta \vee \chi)$	Taut, 1
	(3)	$(\theta \to (\theta \vee \chi))$	CP, 2

We get the proof of Example 1.2 when θ is the formula $x = y$ and χ is the formula $y = \mathbf{0}$. If we make a different choice for θ and χ we obtain a different formal proof, as in the next example.

Example 1.3

1	(1)	$\exists x\, x = \mathbf{0}'$	Ass
1	(2)	$(\exists x\, x = \mathbf{0}' \vee \forall x\, (x + \mathbf{0}) = x)$	Taut, 1
	(3)	$(\exists x\, x = \mathbf{0}' \to (\exists x\, x = \mathbf{0}' \vee \forall x\, (x + \mathbf{0}) = x))$	CP, 2

This formal proof has been obtained by substituting in the table above the formula $\exists x\, x = \mathbf{0}'$ for θ and the formula $\forall x\, (x + \mathbf{0}) = x$ for χ.

◆

Although the formal proofs of Examples 1.2 and 1.3 are different proofs, the fact that they can both be obtained from the above table by making substitutions for θ and χ brings out the fact that they both have the same logical form. The advantage of the above table is that it enables us to concentrate on what is essential, namely that the formulas in the proof are built up from the subformulas θ and χ using connectives in the way shown. The internal structure of θ and χ is not relevant here, and it helps not to show it.

We call such a table a *schematic proof*. Whatever formulas are substituted for the Greek letters, it becomes a valid formal proof.

In propositional logic, which was mentioned at the end of Subsection 3.1 of *Unit 4*, schematic proofs are isolated for study by themselves.

We now give some more examples of schematic proofs.

Example 1.4

1	(1)	ψ	Ass
2	(2)	$\neg\psi$	Ass
1, 2	(3)	ϕ	Taut, 1, 2
1	(4)	$(\neg\psi \to \phi)$	CP, 3

This shows that whatever formulas we substitute for ϕ and ψ, the resulting formula $(\neg\psi \to \phi)$ may be derived from the assumption ψ.

The use of the Tautology Rule on line 3 is justified because the formula $((\psi \,\&\, \neg\psi) \to \phi)$ is always a tautology, as you may readily check. The assumptions in force on line 3 include $\neg\psi$ and hence, by using the Conditional Proof Rule, we can introduce the formula $(\neg\psi \to \phi)$ on line 4; the assumptions in force on this line are those in force on line 3 other than $\neg\psi$.

◆

Example 1.5

1	(1)	ϕ	Ass
2	(2)	ψ	Ass
1, 2	(3)	$(\phi \,\&\, \psi)$	Taut, 1, 2
1	(4)	$(\psi \rightarrow (\phi \,\&\, \psi))$	CP, 3
	(5)	$(\phi \rightarrow (\psi \rightarrow (\phi \,\&\, \psi)))$	CP, 4

The use of the Tautology Rule on line 3 is justified because $((\phi \,\&\, \psi) \rightarrow (\phi \,\&\, \psi))$ is always a tautology. On line 4 we have used the Conditional Proof Rule to drop ψ from the list of assumptions in force, and on line 5 we have used it to drop ϕ from the list of assumptions. Thus we end up with a schematic proof which shows that, for all formulas ϕ and ψ, there is a formal proof of the formula $(\phi \rightarrow (\psi \rightarrow (\phi \,\&\, \psi)))$ which depends on no assumptions.

◆

Example 1.5 provides an example of a common use of the Conditional Proof Rule. Suppose we wish to derive an implication of the form $(\theta \rightarrow \chi)$ from a given set of assumptions. We add the formula θ to the list of assumptions, and then we aim to find a proof of χ from this augmented list of assumptions. If we succeed in this, we can then use the Conditional Proof Rule to derive $(\theta \rightarrow \chi)$ from the remaining assumptions. In Example 1.5 we used this strategy twice. To prove $(\phi \rightarrow (\psi \rightarrow (\phi \,\&\, \psi)))$ from no assumptions, we first made the assumption ϕ with the aim of deriving $(\psi \rightarrow (\phi \,\&\, \psi))$. Then we used the same strategy to derive $(\psi \rightarrow (\phi \,\&\, \psi))$: we added ψ to the list of assumptions and then aimed to derive $(\phi \,\&\, \psi)$. It was straightforward to achieve this latter aim, as we did on line 3, using the Tautology Rule. We then used the Conditional Proof Rule twice to complete the task of deriving $(\phi \rightarrow (\psi \rightarrow (\phi \,\&\, \psi)))$ from no assumptions.

Problem 1.1

The following schematic proofs are incomplete because the assumption numbers on the left-hand side of each line are missing.

(i) Fill in these missing assumption numbers.

(ii) Where the Tautology Rule has been applied, state which tautology has been used.

(a)	(1)	ψ	Ass
	(2)	ϕ	Ass
	(3)	$(\phi \vee \psi)$	Taut, 1
	(4)	$(\phi \,\&\, \psi)$	Taut, 1, 2
	(5)	$((\phi \vee \psi) \,\&\, (\phi \,\&\, \psi))$	Taut, 3, 4
	(6)	$(\psi \rightarrow ((\phi \vee \psi) \,\&\, (\phi \,\&\, \psi)))$	CP, 5

(b)	(1)	$(\phi \vee \chi)$	Ass
	(2)	$\neg\chi$	Ass
	(3)	ϕ	Taut, 1, 2
	(4)	ψ	Ass
	(5)	$(\phi \,\&\, \psi)$	Taut, 3, 4
	(6)	$(\neg\chi \rightarrow (\phi \,\&\, \psi))$	CP, 5
	(7)	$(\chi \vee (\phi \,\&\, \psi))$	Taut, 6
	(8)	$(\psi \rightarrow (\chi \vee (\phi \,\&\, \psi)))$	CP, 7

Problem 1.2

(a) Give a schematic proof to show that for all formulas ϕ, and ψ there is a formal proof of the formula $\neg(\phi \to \psi)$ from the assumptions ϕ and $\neg\psi$.

(b) Show how your schematic proof of part (a) may be extended to give a schematic proof of

$$(\phi \to (\neg\psi \to \neg(\phi \to \psi)))$$

depending on no assumptions.

Problem 1.3

Give schematic proofs to show that for all formulas ϕ and ψ there is a formal proof of each of the following formulas depending on no assumptions.

(a) $(\phi \to ((\phi \to \psi) \to \psi))$

(b) $((\neg\phi \to \neg\psi) \to ((\neg\phi \to \psi) \to \phi))$

You have now seen three rules of proof: the Assumption Rule, the Tautology Rule and the Conditional Proof Rule. Before we go on to look at further rules in Sections 2 and 3, it is worth pausing to reflect on the Tautology Rule. This rule is based on the idea of *tautological* consequence. However, since the aim of our rules is to provide logically valid proofs, a concept based on the idea of *logical* consequence, would it not be better to replace the Tautology Rule with one based on this concept? So, suppose that, in Definition 1.2, we were to replace 'tautological consequence' by 'logical consequence'. The amended rule would still meet our requirement for logical validity. However, it would not meet our requirement for machine checkability. This is because the notion of logical consequence refers to *all possible* interpretations of our formal language, and there might be infinitely many of these. For this reason alone, you might find it plausible that such a rule would not be machine-checkable. In fact it is not even machine-checkable when applied solely to the standard interpretation $\mathscr{N}$, the cause of which failure is the inclusion of the infinite set $\mathbb{N}$ in $\mathscr{N}$. Thus we must settle for the Tautology Rule as given in Definition 1.2, which we have seen is machine-checkable.

A proof that logical consequence is, in general, not machine-checkable will be given in *Unit 8*.

It may now surprise you, having seen that logical validity is linked via logical consequence to all possible interpretations of our formal language, that, by adding just a few more rules, we can obtain a machine-checkable formal system adequate to provide a formal proof of ψ from assumptions $\phi_1, \phi_2, \ldots, \phi_k$ whenever ψ is a logical consequence of these formulas. But such rules do exist and we begin describing them in the next section.

2 QUANTIFIER LOGIC: THE UNIVERSAL QUANTIFIER

In this section and the next we shall describe the rules of proof that we use to handle the quantifiers. We shall deal in this section with the universal quantifier $\forall$, and in the next section with the existential quantifier $\exists$. We shall start this section with an important technical discussion about variables in formulas.

Plainly a major requirement of our rules of proof is that they should handle quantifiers. Hence we shall sometimes refer to the proof system developed in this unit and in *Unit 6* as *quantifier logic*.

2.1 Free and bound variables

In this subsection we shall discuss a distinction between two different roles that variables can play in formulas. The distinction we make is not difficult to understand in general terms, as it corresponds to different ways in which variables are used in everyday mathematics. However, for our purposes, a general idea of the distinction is not quite good enough. We shall need to look in detail at the syntax of formulas so that we have an algorithmic method for making the distinction which concerns us.

We have already alluded to this distinction in *Unit 4*. At the end of Subsection 3.2 we considered the following three formulas

(a) $\exists x\,(x \cdot x) = \mathbf{0}''''$

(b) $\forall y\,\exists x\,(x \cdot x) = y$

(c) $\exists x\,(x \cdot x) = y$

and we asked whether or not they are true in the standard interpretation $\mathscr{N}$. We remarked that formula (a) is true and formula (b) is false, but we cannot say whether or not formula (c) is true unless we know which element of the domain the variable y is interpreted as referring to. This distinction between the role of the variable y in formula (c) and that of the other variables arises from the syntax of these formulas. It has nothing to do with the properties of $\mathscr{N}$. It arises in any interpretation. For example, in the interpretation whose domain is the set $\mathbb{R}$ of real numbers, and with the standard operations of addition and multiplication on this set, formula (c) is true if and only if the variable y is interpreted as referring to an element of $\mathbb{R}$ which has a square root in $\mathbb{R}$, that is, to a non-negative real number.

In formula (a) the variable x is associated with an existential quantifier. Its role is to express the existence of a solution of the equation

$$(x \cdot x) = \mathbf{0}''''$$

In any given interpretation the term $\mathbf{0}''''$ will be interpreted as referring to a specific element of the domain. Either this equation has a solution for this interpretation of $\mathbf{0}''''$ or not. In the former case the formula (a) is true in the given interpretation and in the latter case it is false.

Similarly, in formula (b) both the variables x and y are associated with quantifiers. The variable y is associated with a universal quantifier and so its role is to express the fact that, in a given interpretation, there exists a solution x to the equation

$$(x \cdot x) = y$$

whichever element of the domain the variable y is taken as referring to. For each specific interpretation this fact will be either true or false, and correspondingly formula (b) will be true or false.

In contrast, the variable y is not associated with a quantifier in formula (c), so it neither expresses universality nor existence. In order to determine whether this formula is true in a given interpretation we need to know to which element of the domain it refers. Note that this situation does not arise for the variable x in this formula, as it is associated with an existential quantifier.

We now show how we can make precise this distinction between the two sorts of roles that variables can play. Variables that are associated with quantifiers are called *bound variables* and those that are not are called *free variables*. As we shall see, free variables are so called because, in a sense to be made more precise later, we are free to substitute values for them.

As we comment at the end of this subsection, an analogous distinction occurs in everyday mathematics, where variables associated with the analogue of quantifiers are often called *dummy variables.*

A key point to note is that the same variable will often occur more than once in a formula, and in such cases some of these occurrences will be associated with quantifiers and some not. For example in the formula

$$(\forall x\, \exists y\, y = x' \,\&\, \exists x\, (x \cdot x) = y)$$

the variable y occurs three times. The first two occurrences are associated with the existential quantifier and are *bound occurrences*, but the third is not associated with any quantifier and is thus a *free occurrence.*

We see from this example that the distinction we need to make is between free and bound occurrences of variables in formulas. We do this by thinking about the formation rules for formulas. You will recall that formulas are built up from atomic formulas using the connectives and quantifiers. In an atomic formula all occurrences of variables are free occurrences. The status of the variables does not change when we introduce connectives. The free occurrences of variables in $\neg\phi$, $(\phi \,\&\, \psi)$, $(\phi \vee \psi)$, $(\phi \rightarrow \psi)$ and $(\phi \leftrightarrow \psi)$ correspond to the free occurrences in ϕ and ψ, and likewise for the bound occurrences. It is the introduction of quantifiers that binds variables. Thus in the formulas $\forall v\, \phi$ and $\exists v\, \phi$, all occurrences of the variable v are bound occurrences.

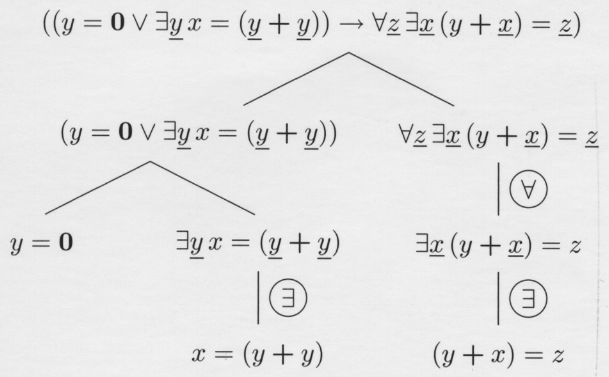
See *Unit 4*, Subsection 2.1.

Thus if we trace how a formula is built up from atomic formulas as we did in *Unit 4*, we can work out which occurrences are free and which are bound.

To illustrate this, let us consider the usual analysis of the formula

$$((y = \mathbf{0} \vee \exists y\, x = (y + y)) \rightarrow \forall z\, \exists x\, (y + x) = z) \qquad (2.1)$$

It is as follows:


$((y = \mathbf{0} \vee \exists \underline{y}\, x = (\underline{y} + \underline{y})) \rightarrow \forall \underline{z}\, \exists \underline{x}\, (y + \underline{x}) = \underline{z})$
$(y = \mathbf{0} \vee \exists \underline{y}\, x = (\underline{y} + \underline{y}))$
$\forall \underline{z}\, \exists \underline{x}\, (y + \underline{x}) = \underline{z}$
∀
$y = \mathbf{0}$
$\exists \underline{y}\, x = (\underline{y} + \underline{y})$
$\exists \underline{x}\, (y + \underline{x}) = z$
∃
∃
$x = (y + y)$
$(y + x) = z$

The underlining indicates bound occurrences of variables.

Formula (2.1) provides another example of the same variable having both free and bound occurrences in a formula. For example, reading from left to right, the first and last occurrences of the variable y are free occurrences, and the other three occurrences of y are bound occurrences.

We can look at this diagram in two ways. Reading from top to bottom, it shows how the original formula can be broken down into components. On the other hand, reading from bottom to top, it shows how this formula is built up from atomic formulas. To distinguish the free and bound occurrences of variables we look at this diagram in this second way. We have labelled the stages at which quantifiers are introduced. Each introduction of a quantifier leads to all the occurrences of the relevant variable being bound in the resulting formula, and these occurrences remain bound as we continue upwards, and hence are bound occurrences in the original formula at the top of the diagram. We have underlined all the bound occurrences of variables from the stage where they first get bound.

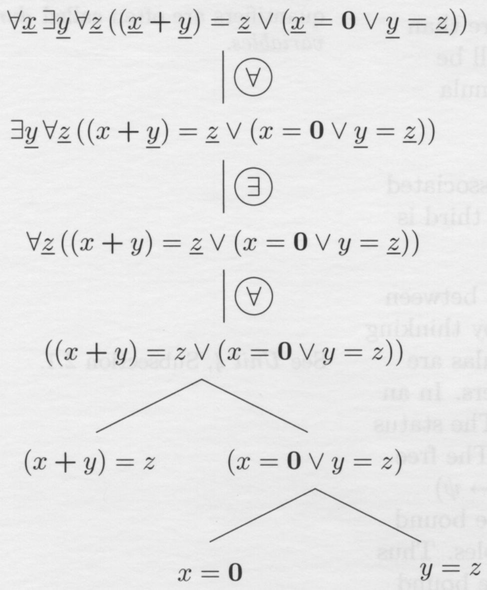
Note that whenever we introduce a quantifier, the occurrence of the variable immediately following the quantifier counts as a bound occurrence. That is why in the example above we have $\exists \underline{x}$, $\exists \underline{y}$ and $\forall \underline{z}$ with the variables underlined.

Clearly the underlining process we have carried out here is purely mechanical, and hence it provides us with an algorithm for deciding whether or not a given occurrence of a variable in a formula is free or bound.

Example 2.1

$$\forall x\, \exists y\, \forall z\, ((x + y) = z \vee (x = \mathbf{0} \vee y = z))$$

The analysis is as follows:


$\forall \underline{x}\, \exists \underline{y}\, \forall \underline{z}\, ((\underline{x} + \underline{y}) = \underline{z} \vee (\underline{x} = \mathbf{0} \vee \underline{y} = \underline{z}))$
$\forall$
$\exists \underline{y}\, \forall \underline{z}\, ((x + \underline{y}) = \underline{z} \vee (x = \mathbf{0} \vee \underline{y} = \underline{z}))$
$\exists$
$\forall \underline{z}\, ((x + y) = \underline{z} \vee (x = \mathbf{0} \vee y = \underline{z}))$
$\forall$
$((x + y) = z \vee (x = \mathbf{0} \vee y = z))$
$(x + y) = z$
$(x = \mathbf{0} \vee y = z)$
$x = \mathbf{0}$
$y = z$

Thus all the occurrences of x, y and z in the formula are bound. If you noticed that the formula has the form $\forall x\, \exists y\, \forall z\, \phi$ you could have drawn this conclusion without the need for the above analysis. ◆

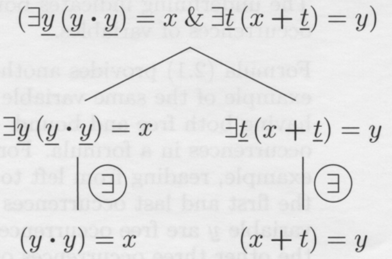
Example 2.2

(a) $(\exists y\, (y \cdot y) = x \,\&\, \exists t\, (x + t) = y)$

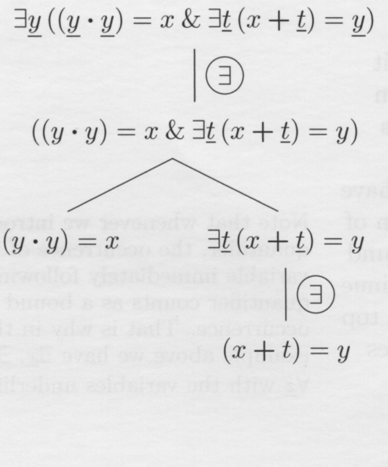
(b) $\exists y\, ((y \cdot y) = x \,\&\, \exists t\, (x + t) = y)$

Here are the analyses

(a)


$(\exists \underline{y}\, (\underline{y} \cdot \underline{y}) = x \,\&\, \exists \underline{t}\, (x + \underline{t}) = y)$
$\exists \underline{y}\, (\underline{y} \cdot \underline{y}) = x$
$\exists \underline{t}\, (x + \underline{t}) = y$
$\exists$
$\exists$
$(y \cdot y) = x$
$(x + t) = y$

(b)


$\exists \underline{y}\, ((\underline{y} \cdot \underline{y}) = x \,\&\, \exists \underline{t}\, (x + \underline{t}) = \underline{y})$
$\exists$
$((y \cdot y) = x \,\&\, \exists \underline{t}\, (x + \underline{t}) = y)$
$(y \cdot y) = x$
$\exists \underline{t}\, (x + \underline{t}) = y$
$\exists$
$(x + t) = y$

The examples show how important bracketing can be. In (a) the first three occurrences of y are bound, whereas the fourth occurrence is free. In (b) all the occurrences of y are bound. In (a) and (b) both occurrences of x are free, and both occurrences of t are bound.

◆

After some practice, it should be possible for you to distinguish between free and bound occurrences of variables without the need to write down the analysis of the formula.

In Example 2.1, there are no free occurrences of variables. Formulas with this property are called *sentences.*

Definition 2.1 Sentence

A *sentence* is a formula in which there are no free occurrences of variables.

Thus a sentence is a formula in which all occurrences of variables are bound occurrences. Note that there are some formulas in which there are no variables at all. One example is $\neg\, 0'' = 0'''$. Such formulas count as sentences. In any given interpretation, a particular sentence is either true or false — there is no extra complication of having to specify values in the domain for any free variables.

To avoid complications with free variables, it is customary to express axioms for number theory, which we shall do in *Unit 6*, as sentences.

We end this subsection with some remarks about free and bound variables in everyday mathematics. In this context formulas are usually expressions which can take numerical values. Here are some examples.

$$\text{(a)} \int_0^1 x^2\,dx \qquad \text{(b)} \int_0^a x^2\,dx \qquad \text{(c)} \sum_{n=0}^{\infty} \frac{1}{n!} \qquad \text{(d)} \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Expression (a) has a definite numerical value, whereas in (b) the value depends on the value of the free variable a. However, in (a) and (b) it does not make sense to ask for the value of x; its role is to show which function we are integrating. We could replace x in (a) or (b) by another variable without changing the meaning. So, for example,

$$\int_0^1 x^2\,dx = \int_0^1 t^2\,dt.$$

In logic we call such a variable a bound variable; in everyday mathematics it is more usual to call it a *dummy variable.* Likewise, in (c) and (d), n is a bound or dummy variable, whereas in (d) x is free.

Problem 2.1

In the following formulas, determine which occurrences of variables are free and which are bound. State which of the formulas are sentences.

(a) $\forall x\, \exists y\, \neg\, x = y$

(b) $((x = y \lor \exists t\,(x + t) = y) \lor \exists t\,(y + t) = x)$

(c) $(\exists z\,(z \cdot x) = y \to \forall y\,(\exists z\,(z \cdot y) = x \to \neg \exists x\,(x \cdot x) = y))$

(d) $\exists x\,(x = y \lor \neg\, x = y)$

(e) $(\exists x\, x = y \lor \neg\, x = y)$

2.2 The Universal Quantifier Elimination Rule

Now that we have dealt with the distinction between free and bound occurrences of variables, we are in a position to begin describing the formal rules of proof for the quantifiers. We start with the universal quantifier $\forall$. We have two rules of proof associated with this symbol, one for eliminating it from a formula and the other for introducing it into one. We consider the elimination rule first.

Our method for devising the quantifier rules will be to look at how quantifiers are handled in informal mathematical arguments, and to mirror this in our formal rules. So we begin by looking at some examples. Consider the following simple argument from elementary number theory.

> From the identity $x^2 - 1 = (x-1)(x+1)$, it follows that $7^2 - 1 = (7-1)(7+1)$ and so is divisible by 8.

At first sight no universal quantifier is present; but it must be remembered that in informal mathematics the symbol $\forall$ is not often used, other methods being used to indicate that statements are universally true. In the example above, the word 'identity' has been used to indicate that the equation is true for all values (in the relevant domain — here, the natural numbers) of the variable x. So the deduction in this passage, expressed slightly more formally, is as follows.

From

$$\forall x\, x^2 - 1 = (x-1)(x+1) \tag{2.2}$$

we deduce that

$$7^2 - 1 = (7-1)(7+1) \tag{2.3}$$

We can see that, in going from (2.2) to (2.3), the universal quantifier $\forall$ and the variable x following it have been eliminated, and all the remaining occurrences of x have been replaced by occurrences of the symbol 7; the variable which expressed generality in (2.2) has been replaced in (2.3) by the symbol for a particular number.

We are not obliged to replace the variable by a numeral; it could be replaced by a more complicated expression. For example, in an argument in trigonometry, where the range of values taken by the variables would be the set of real numbers, we might want to deduce from (2.2) that

$$(\sin t)^2 - 1 = (\sin t - 1)(\sin t + 1). \tag{2.4}$$

So this time x has been replaced by $\sin t$, and it is easy to imagine situations where we would want to substitute even more complicated expressions for x. In general, the sort of expressions we might want to substitute for x will be those that can stand for elements of the domain of the interpretation (usually numbers). Recall that, in *Unit 4*, we called these expressions *terms*. So the rule we are looking for will be one which allows us to drop a universal quantifier and the variable which follows it from the beginning of a formula, and to replace all the remaining occurrences of the variable, which are now free occurrences, by a given term. In order to be able to state this rule succinctly, it is convenient to introduce some notation.

Definition 2.2

If ϕ is a formula, v is a variable and τ is a term, then $\phi(\tau/v)$ is the result of substituting the term τ for each free occurrence of v in ϕ.

You can think of $\phi(\tau/v)$ as meaning 'ϕ with τ replacing all the free occurrences of v'.

It should not be difficult to convince yourself that if ϕ is a formula then so is $\phi(\tau/v)$.

Example 2.3

(a) Let ϕ be the formula

$$\exists y\, y = (x + x)$$

Both occurrences of x in ϕ are free occurrences and so $\phi(\mathbf{0}/x)$ is the formula

$$\exists y\, y = (\mathbf{0} + \mathbf{0})$$

and $\phi((x \cdot z)/x)$ is the formula

$$\exists y\, y = ((x \cdot z) + (x \cdot z))$$

(b) Let ϕ be the formula

$$(\forall z\, (y \cdot z) = y \to \forall y\, (y \cdot z) = y)$$

Here the first two occurrences of the variable y are free occurrences, but the remaining occurrences are all bound. Hence the formula $\phi(\mathbf{0}'/y)$ is

$$(\forall z\, (\mathbf{0}' \cdot z) = \mathbf{0}' \to \forall y\, (y \cdot z) = y)$$

For a similar reason $\phi(\mathbf{0}'/z)$ is

$$(\forall z\, (y \cdot z) = y \to \forall y\, (y \cdot \mathbf{0}') = y)$$

♦

Problem 2.2

In each of the following cases, where you are given a formula ϕ, a term τ and a variable v, write down the formula $\phi(\tau/v)$.

(a) ϕ is $\exists y\, (y \cdot y) = (x \cdot (x + y))$, τ is $(z'' + \mathbf{0})$, v is x.

(b) ϕ is $(\exists x\, (x \cdot x) = y \,\&\, x = (z + y))$, τ is $\mathbf{0}'$, v is x.

(c) ϕ is $\exists x\, ((x \cdot x) = y \,\&\, x = (z + y))$, τ is $\mathbf{0}'$, v is x.

(d) ϕ is $\exists x\, ((x \cdot x) = y \,\&\, x = (z + y))$, τ is $\mathbf{0}'$, v is y.

(e) ϕ is $\exists y\, y = (x + x)$, τ is y', v is x.

(f) ϕ is $\exists x\, (z + x) = (z \cdot x)$, τ is x, v is z.

(g) ϕ is $\exists x\, (z + x) = (z \cdot x)$, τ is z, v is z.

In terms of the notation we have just introduced, the formal rule we are aiming at should tell us that from a formula $\forall v\, \phi$ we can derive $\phi(\tau/v)$, where τ is some term.

Before we state this rule we need first to check that it meets our requirements of logical validity and machine checkability. Since we have an algorithm to decide which occurrences of variables in a formula are free occurrences, it is easily seen that the second requirement is met.

At first sight, everything also seems to be in order with the requirement of logical validity. If the formula $\forall v\, \phi$ is true in some interpretation, then every element of the domain must have the property expressed by the formula ϕ. In particular, the element corresponding to term τ has this property, and so it would seem that the formula $\phi(\tau/v)$ must also be true. This is almost correct, but it overlooks a technical complication which we now explain.

When we were discussing the difference between free and bound occurrences of variables, we said that the truth of the formula depends on the interpretation of the free variables, and in this sense expresses properties of these variables. The bound variables are just 'dummies' whose role is, for example, to express generality. The deduction of $\phi(\tau/v)$ from $\forall v\, \phi$ will be valid only if $\phi(\tau/v)$ says the same thing about τ as ϕ says about v. Things will go wrong if there are variables in τ which become bound when we substitute τ for the free occurrences of v in ϕ.

For example, let ϕ be the formula $\exists y\, y = (x + x)$, so that $\forall x\, \phi$ is the formula

$$\forall x\, \exists y\, y = (x + x) \tag{2.5}$$

and let τ be the term y', so that $\phi(\tau/x)$ is the formula

$$\exists y\, y = (y' + y') \tag{2.6}$$

Formula (2.5) is true in the standard interpretation $\mathscr{N}$: it says that, for each natural number, there is another natural number which is twice the original number. However, formula (2.6) is false in $\mathscr{N}$: it asserts that there is a natural number y such that $y = (y + 1) + (y + 1)$, which is clearly not true. Since there is an interpretation which makes (2.5) true but (2.6) false, we cannot validly deduce (2.6) from (2.5). The reason that something has gone wrong is that the term τ contains the variable y, and these occurrences of y become bound when we substitute τ for x in ϕ.

To avoid this sort of problem and ensure that the inference of $\phi(\tau/v)$ from $\forall v\, \phi$ is logically valid, we shall require that no occurrences of variables in τ become bound when we substitute τ for free occurrences of v in ϕ. It is convenient to have a shorthand expression for this condition, so we make the following definition.

Definition 2.3 Free substitution

We say that the term τ may be *freely substituted* for the variable v in the formula ϕ if no occurrences of variables in τ become bound when we substitute τ for free occurrences of v in ϕ.

Example 2.4

(a) Let ϕ be the formula

$$\exists y\, (y \cdot y) = (x \cdot (x + y))$$

and let τ be the term $(z'' + \mathbf{0})$. The only variable in τ is z and, since there are no quantifiers involving z in ϕ, neither of the occurrences of z in $\phi(\tau/x)$ is bound. So τ may be freely substituted for x in ϕ.

(b) Let ϕ be the formula

$$\exists z\, (z \cdot y) = (x \cdot (z + y))$$

and let τ be the term $(z'' + \mathbf{0})$. The term τ may *not* be freely substituted for x in ϕ since the variable z occurring in τ becomes bound in $\phi(\tau/x)$, which is the formula

$$\exists z\, (z \cdot y) = ((z'' + \mathbf{0}) \cdot (z + y))$$

(c) Let ϕ be the formula

$$(x = \mathbf{0}' \vee \exists x\, y = x')$$

and let τ be the term $(x + y)$. Only the first occurrence of x in ϕ is a free occurrence and hence $\phi(\tau/x)$ is the formula

$$((x + y) = \mathbf{0}' \vee \exists x\, y = x')$$

We see that both the variables x and y which occur in τ gives rise to free occurrences of these variables in $\phi(\tau/x)$. Thus τ may be freely substituted for x in ϕ.

However, τ may *not* be freely substituted for y in ϕ, since the formula $\phi(\tau/y)$ is

$$(x = \mathbf{0}' \vee \exists x\, (x + y) = x')$$

and the occurrence of the variable x in the term $(x + y)$ becomes bound in $\phi(\tau/y)$. ◆

Problem 2.3

In which of the following cases may the term τ be freely substituted for the variable v in the formula ϕ?

(a) ϕ is $(\exists z\, z = y' \,\&\, \exists t\, (t \cdot t) = x)$, τ is $(z'' \cdot y)$, v is x.

(b) ϕ is $\exists z\, (z = y' \,\&\, \exists t\, (t \cdot t) = x)$, τ is $(z'' \cdot y)$, v is x.

(c) ϕ is $(\forall x\, \exists z\, x = (z + z) \rightarrow \exists t\, (t + t) = (y + z))$, τ is $(y + x)$, v is y.

(d) ϕ is $\forall x\, \exists y\, ((y + y) = x \vee (y + y') = x)$, τ is $(y + x)$, v is y.

(e) ϕ is $\forall y\, (\exists x\, (x + x) = y \rightarrow y = (x' + x))$, τ is x, v is x.

We can now state our formal rule for the elimination of the universal quantifier.

Definition 2.4 Universal Quantifier Elimination Rule (UE)

If the formula $\forall v\, \phi$ occurs on a line in a formal proof and τ is a term which may be freely substituted for the variable v in ϕ, then on any subsequent line we may introduce the formula $\phi(\tau/v)$, which will depend on the same assumptions as does $\forall v\, \phi$.

With the specified condition on the term τ, the rule can be shown to be logically valid. Any algorithm to check which occurrences of a variable v are free can be extended to check whether a term τ may be freely substituted for these occurrences. So our requirements for logical validity and machine checkability are met.

Example 2.5

We cannot give any very interesting examples of the use of the Universal Quantifier Elimination Rule until we have some other rules for quantifiers at our disposal. Meanwhile, here is a simple example of a formal proof which uses this rule twice.

1	(1)	$\forall x\, \forall y\, (x + y) = (y + x)$	Ass
1	(2)	$\forall y\, ((z + x) + y) = (y + (z + x))$	UE, 1
1	(3)	$((z + x) + x) = (x + (z + x))$	UE, 2

If we call the formula on line 1 $\forall x\, \phi$, so that ϕ is the formula $\forall y\, (x + y) = (y + x)$, then the formula on line 2 is $\phi((z + x)/x)$. The use of the the UE Rule in going from line 1 to line 2 is legitimate as the term $(z + x)$ may be freely substituted for x in ϕ. If now we call the formula on line 2 $\forall y\, \psi$, so that ψ is the formula $((z + x) + y) = (y + (z + x))$, then the formula on line 3 is $\psi(x/y)$. Again, the use of the UE Rule is legitimate, since the term x may be freely substituted for y in ψ. Indeed, as there are no quantifiers in ψ, any term may be freely substituted for y in ψ. ♦

Problem 2.4

Show that the formula

$$(\mathbf{0}' + \mathbf{0}'') = (\mathbf{0}'' + \mathbf{0}')$$

can be derived from the assumption

$$\forall x\, \forall y\, (x + y) = (y + x)$$

Problem 2.5

Is it the case that, for any term τ, the formula $\exists y\, \neg\, \tau = y$ is a logical consequence of the formula $\forall x\, \exists y\, \neg\, x = y$?

The last problem is a reminder of why we need to be careful about the details of substituting a term for a variable within a formula. We shall need to be similarly careful with several of our remaining rules.

2.3 The Universal Quantifier Introduction Rule

To find the appropriate rule for introducing universal quantifiers into formal proofs, we need to consider how *universal statements*, that is, statements or formulas of the form $\forall v\, \phi$, are proved in mathematics.

Note that, in more advanced books on logic, the term *universal formula* is used to mean something stronger than one of the form $\forall v\, \phi$.

Consider, for example, the proof of the following theorem about groups.

Theorem

Let G be a group. Then for all g, h in G,

$$(gh)^{-1} = h^{-1}g^{-1}. \tag{2.7}$$

Don't worry if you are not familiar with group theory; the mathematical details of the proof are not important for our purposes.

Proof

Take g, h in G. Then

$$(h^{-1}g^{-1})(gh) = h^{-1}(g^{-1}g)h = h^{-1}eh = h^{-1}h = e,$$

where e is the identity element of G.

Thus $h^{-1}g^{-1}$ is the inverse of gh, that is,

$$(gh)^{-1} = h^{-1}g^{-1}. \tag{2.8}$$

This completes the proof. ■

This proof establishes the universal statement (2.7) about all elements of the group G. We proved it by taking two elements g and h of G and showing that they satisfy formula (2.8). Why does this enable us to claim that we have thereby proved the universal statement (2.7) and hence that we have completed the proof of the theorem? The reason is that in deriving (2.8) we made no special assumptions about g and h. We used several properties of groups (without mentioning them explicitly) such as the associative property, but all we assumed about g and h was that they were elements of G. That is why the universal statement (2.7) follows from (2.8).

We see from this that the standard method for proving a universal formula such as $\forall v\, \phi$ is to prove ϕ without making any special assumptions about v. It then follows that ϕ is true for every v being considered, that is, $\forall v\, \phi$ is true. We saw earlier that a formula tells us something about a variable v only when v occurs freely in the formula. Thus, a derivation of ϕ which makes no special assumptions about v corresponds to a formal proof of ϕ from assumption formulas which contain no free occurrences of v. This leads us to the following rule.

> ***Definition 2.5 Universal Quantifier Introduction Rule (UI)***
>
> If the formula ϕ occurs on a line of a formal proof and the variable v has no free occurrences in any of the assumptions in force on that line, then on any subsequent line we may introduce the formula $\forall v\, \phi$, which will depend on the same assumptions as does ϕ.

With the restriction that the assumptions contain no free occurrences of v, the rule can be shown to be logically valid. As with the Universal Quantifier Elimination Rule, it is easy to see that this rule is machine-checkable.

Example 2.6

The following formal proof is typical in the sense that we make several applications of the Universal Quantifier Elimination Rule before we are ready to apply the the Universal Quantifier Introduction Rule. In this example, each time we apply the UE Rule, the term τ is just the variable x itself.

1	(1)	$\forall x \forall y \forall z\,(x + (y + z)) = ((x + y) + z)$	Ass
1	(2)	$\forall y \forall z\,(x + (y + z)) = ((x + y) + z)$	UE, 1
1	(3)	$\forall z\,(x + (x + z)) = ((x + x) + z)$	UE, 2
1	(4)	$(x + (x + x)) = ((x + x) + x)$	UE, 3
1	(5)	$\forall x\,(x + (x + x)) = ((x + x) + x)$	UI, 4

To see whether the use of the UI Rule on line 5 is legitimate, we need to look at the line from which line 5 has been derived. The annotation to the right of this line tells us that is has been derived from line 4 using the UI Rule. For this use to be legitimate, the variable x must have no free occurrences in any of the assumptions on which line 4 depends. The only assumption in force on line 4 is the formula on line 1. We see that x has no free occurrences in this formula. So the use of the UI Rule on line 5 is legitimate. The UI Rule then tells us that line 5 depends on exactly the same assumptions as does line 4.

◆

Problem 2.6

Show that the formula

$$\forall x\,((x + x) \cdot x) = ((x \cdot x) + (x \cdot x))$$

can be derived from the assumption

$$\forall x \forall y \forall z\,((x + y) \cdot z) = ((x \cdot z) + (y \cdot z))$$

Example 2.7

We shall show that the formula

$$\forall y \forall x\,(x + y) = (y + x)$$

can be derived from the assumption

$$\forall x \forall y\,(x + y) = (y + x)$$

1	(1)	$\forall x \forall y\,(x + y) = (y + x)$	Ass
1	(2)	$\forall y\,(x + y) = (y + x)$	UE, 1
1	(3)	$(x + y) = (y + x)$	UE, 2
1	(4)	$\forall x\,(x + y) = (y + x)$	UI, 3
1	(5)	$\forall y \forall x\,(x + y) = (y + x)$	UI, 4

On line 4 we use the UI Rule to add the prefix $\forall x$ to the formula on line 3. To see whether this is a legitimate use of the UI Rule, we must check that the variable x does not occur freely in any of the assumptions on which line 3 depends. Similarly, to see whether the use of the UI Rule on line 5 is legitimate, we must check that the variable y does not occur freely in any of the assumptions on which line 4 depends. Both line 3 and line 4 depend on the formula on line 1 as an assumption. The formula on line 1 is a sentence, that is, it contains no free occurrences of variables. Hence both uses of the UI Rule in this proof are legitimate.

◆

On line 2 of the formal proofs in both Examples 2.6 and 2.7, we have used a particularly simple case of the UE Rule. In each case the term τ is the same as the variable v it replaces, so $\phi(\tau/v)$ is $\phi(v/v)$; and, clearly, $\phi(v/v)$ is identical to ϕ. So this simple case of the UE Rule can be stated as:

> from $\forall v\, \phi$ we can derive ϕ, which will depend on the same assumptions as does $\forall v\, \phi$.

Since v can certainly be freely substituted for v in ϕ, this is always a legitimate use of the UE Rule.

It is convenient at this stage to introduce the following notation.

Definition 2.6

We write

$$\phi_1, \phi_2, \ldots, \phi_k \vdash \psi$$

to indicate that the formula ψ can be derived from assumptions included among the formulas $\phi_1, \phi_2, \ldots, \phi_k$.

Thus, if $\phi_1, \phi_2, \ldots, \phi_k$ and ψ are formulas, we may write

$$\phi_1, \phi_2, \ldots, \phi_k \vdash \psi$$

if there is a formal proof on whose last line is the formula ψ and the assumptions in force on that line are included among $\phi_1, \phi_2, \ldots, \phi_k$. In particular, we may write

$$\vdash \psi$$

if there is a formal proof of the formula ψ depending on no assumptions. The symbol $\vdash$ is often called the *turnstile symbol*.

The turnstile symbol was introduced into logic by Frege. It is *not* a symbol of our formal language.

In terms of this new notation we can say that the formal proof of Example 2.7 shows that

$$\forall x\, \forall y\, (x+y) = (y+x) \vdash \forall y\, \forall x\, (x+y) = (y+x)$$

It is not difficult to see that this is a particular instance of the fact that, for every formula ϕ and all variables u and v,

$$\forall u\, \forall v\, \phi \vdash \forall v\, \forall u\, \phi$$

Example 2.8

For all formulas ϕ and ψ,

$$\forall v\, \phi, \forall v\, (\phi \to \psi) \vdash \forall v\, \psi$$

Note first that, because here we have made a general claim about the existence of a formal proof whatever the formulas ϕ and, ψ are, we shall need to justify it by giving a schematic proof as described in Subsection 1.2. We seek a schematic proof in which $\forall v\, \psi$ occurs on the last line and the assumptions in force on that line are $\forall v\, \phi$ and $\forall v\, (\phi \to \psi)$. Thus our first move will be to introduce these two formulas as assumptions.

We are aiming to derive the conclusion $\forall v\,\psi$ and we know that if we can derive ψ then, under the right conditions, we can derive $\forall v\,\psi$ by the UI Rule. Having introduced $\forall v\,\phi$ and $\forall v\,(\phi \to \psi)$ as assumptions, it is natural to use the UE Rule to derive ϕ and $(\phi \to \psi)$ from these assumptions. In fact, we can use the special case of the UE Rule mentioned immediately after Example 2.7. The gap we then have to fill is to go from ϕ, $(\phi \to \psi)$ to the desired conclusion ψ. But we have seen that the formula $((\phi \mathbin{\&} (\phi \to \psi)) \to \psi)$ is a tautology. Thus we can derive ψ from ϕ, $(\phi \to \psi)$ by the Tautology Rule. Let us see if we can carry out this plan.

See the solution to Problem 3.7 of *Unit 4*.

1	(1)	$\forall v\,\phi$	Ass
2	(2)	$\forall v\,(\phi \to \psi)$	Ass
1	(3)	ϕ	UE, 1
2	(4)	$(\phi \to \psi)$	UE, 2
1, 2	(5)	ψ	Taut, 3, 4
1, 2	(6)	$\forall v\,\psi$	UI, 5

The use of the UI Rule is legitimate since the variable v does not have any free occurrences in either of the assumptions on which line 5 depends, namely the formulas on lines 1 and 2.

◆

There are no infallible rules for finding formal proofs, but it is often possible to formulate a plan, as in the preamble to the schematic proof in Example 2.8. The next example involves another helpful tip.

We shall summarize some guidelines for finding formal proofs in Section 1 of *Unit 6*, after we have met all the quantifier rules.

Example 2.9

For all formulas ϕ and ψ,

$$\forall v\,(\phi \to \psi) \vdash (\forall v\,\phi \to \forall v\,\psi)$$

Since we wish to show that we can derive the formula $(\forall v\,\phi \to \forall v\,\psi)$ from the assumption $\forall v\,(\phi \to \psi)$, we introduce this latter formula as an assumption on the first line. We then think about how we might derive the conclusion $(\forall v\,\phi \to \forall v\,\psi)$. If we can derive $\forall v\,\psi$ from assumptions which include $\forall v\,\phi$ then, by using the Conditional Proof Rule, we are able to derive $(\forall v\,\phi \to \forall v\,\psi)$ from the remaining assumptions. So we introduce $\forall v\,\phi$ as a second assumption. We are now aiming to derive $\forall v\,\psi$. This puts us in exactly the same situation as in Example 2.8 and hence we use the same method as in that example. We thus arrive at the following formal proof.

1	(1)	$\forall v\,(\phi \to \psi)$	Ass
2	(2)	$\forall v\,\phi$	Ass
1	(3)	$(\phi \to \psi)$	UE, 1
2	(4)	ϕ	UE, 2
1, 2	(5)	ψ	Taut, 3, 4
1, 2	(6)	$\forall v\,\psi$	UI, 5
1	(7)	$(\forall v\,\phi \to \forall v\,\psi)$	CP, 6

You will note that, except for a change in the order, the first six lines are exactly the same as those of the schematic proof of Example 2.8. Then a use of the Conditional Proof Rule gives us the formula $(\forall v\,\phi \to \forall v\,\psi)$ at which we are aiming.

◆

In Example 2.9 we have used the following standard technique. If we want to derive an implication, say $(\theta \to \chi)$, from a certain set of assumptions, then we add θ to these assumptions. We then try to derive χ. If we succeed in doing this, then, by using the Conditional Proof Rule, we obtain a derivation of $(\theta \to \chi)$ from the remaining assumptions.

Problem 2.7

Show that, for all formulas ϕ and ψ,

$$\forall v\,(\phi \,\&\, \psi) \vdash (\forall v\,\phi \,\&\, \forall v\,\psi)$$

In the final problem of the section we ask you to show that the UI Rule would not be logically valid if the condition that the variable v has no free occurrences in any relevant assumptions is dropped.

Problem 2.8

Give an example of formulas ϕ and $\phi_1, \phi_2, \ldots, \phi_k$ for which ϕ is a logical consequence of $\phi_1, \phi_2, \ldots, \phi_k$, for which the variable v occurs freely in at least one of the ϕ_i (thus disobeying the condition in the UI Rule), and such that $\forall v\,\phi$ is *not* a logical consequence of $\phi_1, \phi_2, \ldots, \phi_k$. *Hint*: Try the formula $v = \mathbf{0}$ for ϕ. Can you think of a single formula ϕ_1, in which v occurs freely, for which this ϕ, but not $\forall v\,\phi$, is a logical consequence of ϕ_1?

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3 QUANTIFIER LOGIC: THE EXISTENTIAL QUANTIFIER

In this section we shall look at the two rules for manipulating the existential quantifier. The first rule will tell us how to infer an *existential formula*, that is, one of the form $\exists v\,\phi$. The second rule will tell us how to make inferences from a statement of this form.

Note that, in more advanced books on logic, the term *existential formula* is used to mean something stronger than one of the form $\exists v\,\phi$.

3.1 The Existential Quantifier Introduction Rule

In informal mathematics the direct method for proving that there exists an object with a certain property is to exhibit a specific example of an object which has that property. Here is a simple case of such a proof.

Theorem

There exists a non-zero 2×2 matrix whose square is the zero matrix.

Proof

Let $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$. Then $\mathbf{A}^2 = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

Thus $\mathbf{A}$ is a non-zero 2×2 matrix with $\mathbf{A}^2$ the zero matrix. ■

The pattern of this argument gives us our formal rule for existential quantifier introduction. If we have a formula $\phi(\tau/v)$ which expresses the fact that a certain term τ has a certain property, then we can deduce from it the formula $\exists v\,\phi$ which expresses the fact that there is some object which has that property. Put the other way round, this says that to derive $\exists v\,\phi$, the direct method is to find a term τ such that the formula $\phi(\tau/v)$, which results from ϕ when we substitute τ for the free occurrences of v, may be derived. As with the Universal Quantifier Elimination Rule, we need to impose the restriction that τ may be freely substituted for v in ϕ.

Thus we are led to the following rule.

> ***Definition 3.1 Existential Quantifier Introduction Rule (EI)***
>
> If τ is a term which may be freely substituted for the variable v in the formula ϕ, and the formula $\phi(\tau/v)$ occurs on a line of a formal proof, then on any subsequent line we may introduce the formula $\exists v\, \phi$, which will depend on the same assumptions as does $\phi(\tau/v)$.

With the specified condition on the term τ, the rule can be shown to be logically valid. It is easy to see that the rule is machine-checkable.

Example 3.1

We show that

$$\exists y\, \mathbf{0}'' = (y+y) \vdash \exists z\, \exists y\, z = (y+y)$$

$$\begin{array}{llll} 1 & (1) & \exists y\, \mathbf{0}'' = (y+y) & \text{Ass} \\ 1 & (2) & \exists z\, \exists y\, z = (y+y) & \text{EI}, 1 \end{array}$$

If we let ϕ be the formula $\exists y\, z = (y+y)$, then the formula on line 1 is $\phi(\mathbf{0}''/z)$ and that on line 2 is $\exists z\, \phi$. Since the term $\mathbf{0}''$ may be freely substituted for z in ϕ, this is a legitimate use of the EI Rule. The assumption in force on line 2 is the same as that on line 1.

◆

It is worth noting that a term which contains no variables may be freely substituted for any free variable in any formula.

Example 3.2

We show that

$$\forall x\, (x+\mathbf{0}) = x \vdash \forall x\, \exists y\, (x+y) = x$$

$$\begin{array}{llll} 1 & (1) & \forall x\, (x+\mathbf{0}) = x & \text{Ass} \\ 1 & (2) & (x+\mathbf{0}) = x & \text{UE}, 1 \\ 1 & (3) & \exists y\, (x+y) = x & \text{EI}, 2 \\ 1 & (4) & \forall x\, \exists y\, (x+y) = x & \text{UI}, 3 \end{array}$$

Note that we cannot use the EI Rule to go directly from line 1 to line 4 since the EI Rule allows us to introduce an existential quantifier only at the beginning of a formula. So we cannot immediately insert $\exists y$ in the middle of the formula on line 1.

If we let ϕ be the formula $(x+y) = x$, then the formula on line 2 is $\phi(\mathbf{0}/y)$ and that on line 3 is $\exists y\, \phi$.

◆

Example 3.3

We show that

$$(\mathbf{0}+\mathbf{0}) = \mathbf{0} \vdash \exists x\, (\mathbf{0}+x) = \mathbf{0}$$

$$\begin{array}{llll} 1 & (1) & (\mathbf{0}+\mathbf{0}) = \mathbf{0} & \text{Ass} \\ 1 & (2) & \exists x\, (\mathbf{0}+x) = \mathbf{0} & \text{EI}, 1 \end{array}$$

Note that the assumption $(\mathbf{0}+\mathbf{0}) = \mathbf{0}$ can be used to derive several different formulas (see Problem 3.1(c) for example).

Here if we let ϕ be the formula $(\mathbf{0}+x) = \mathbf{0}$, then the formula on line 1 is $\phi(\mathbf{0}/x)$ and that on line 2 is $\exists x\, \phi$.

◆

Problem 3.1

Show the following.

(a) $\forall x\,(x + \mathbf{0}) = x \vdash \exists y\,\forall x\,(x + y) = x$

(b) $\forall x\,(x \cdot \mathbf{0}') = x \vdash \forall x\,\exists y\,(x \cdot y) = x$

(c) $(\mathbf{0} + \mathbf{0}) = \mathbf{0} \vdash \exists y\,(y + y) = \mathbf{0}$

(d) $\forall x\,(x + \mathbf{0}) = x \vdash \exists x\,(x + x) = x$

3.2 The Existential Hypothesis Rule

The fourth rule of our formal system for handling the quantifiers is rather more subtle than the other three. By analogy with the rules for the universal quantifier, you might expect there to be some sort of existential quantifier elimination rule which would enable us to drop an initial $\exists v$ from the beginning of a formula. But deductions of this kind would not be valid. From the fact that there is *some* object with a given property we cannot deduce that *any* object has that property. So any rule that enabled us to derive $\phi(\tau/v)$ from $\exists v\,\phi$ could not possibly be logically valid.

To discover what the second existential quantifier rule should be, we need to consider how we make deductions from existential statements in informal mathematics. The example we have chosen to look at again comes from the theory of matrices. We shall begin by giving a simple proof, just as you might find in an algebra textbook. Then, in order to make the logical form of the argument clearer, we rewrite the proof using the logical symbols. Finally we shall extract from this the logical rule we are seeking. Here then is our theorem and proof.

Theorem

If the matrix $\mathbf{A}$ has an inverse, then $\det \mathbf{A} \neq 0$.

Recall that $\det \mathbf{A}$ stands for the 'determinant of $\mathbf{A}$'.

Proof

Suppose that $\mathbf{B}$ is the inverse of $\mathbf{A}$. Then $\mathbf{AB} = \mathbf{I}$, where $\mathbf{I}$ is the identity matrix. Therefore $\det \mathbf{A} \times \det \mathbf{B} = \det \mathbf{AB} = \det \mathbf{I} = 1$. Hence $\det \mathbf{A} \neq 0$. ■

Now let us symbolize '$\mathbf{B}$ is the inverse of $\mathbf{A}$' by '$\mathbf{B}\,\mathrm{inv}\,\mathbf{A}$'. The theorem asserts that from $\exists\mathbf{B}\,\mathbf{B}\,\mathrm{inv}\,\mathbf{A}$ it follows that $\det \mathbf{A} \neq 0$. The proof, however, begins 'Suppose that $\mathbf{B}\,\mathrm{inv}\,\mathbf{A}$', then, after some algebra, we reach the conclusion 'Hence $\det \mathbf{A} \neq 0$'. Thus, although the theorem asserts that $\det \mathbf{A} \neq 0$ follows from the assumption that $\exists\mathbf{B}\,\mathbf{B}\,\mathrm{inv}\,\mathbf{A}$, in the proof we do not start from this assumption. Instead, the assumption with which we start the proof is $\mathbf{B}\,\mathrm{inv}\,\mathbf{A}$. However, when we have managed to deduce from this that $\det \mathbf{A} \neq 0$, we claim that the proof has been completed, that is, that we have in fact shown that $\det \mathbf{A} \neq 0$ follows from the assumption $\exists\mathbf{B}\,\mathbf{B}\,\mathrm{inv}\,\mathbf{A}$. The reason we can make this claim is that, in the course of the proof, we made no assumption about $\mathbf{B}$ other than that given by our assumption $\mathbf{B}\,\mathrm{inv}\,\mathbf{A}$, and in the conclusion that we reach, that $\det \mathbf{A} \neq 0$, there is no mention of $\mathbf{B}$. So this conclusion really depends only on the assumption that there is *some* matrix $\mathbf{B}$ which is the inverse of $\mathbf{A}$, that is, the conclusion really depends only on the assumption that $\exists\mathbf{B}\,\mathbf{B}\,\mathrm{inv}\,\mathbf{A}$, which is what the theorem asserts.

This example is typical of the way existential hypotheses are used in mathematical arguments. To deduce some conclusion from the assumption that there is some object v which satisfies the property expressed by ϕ, that is, from the assumption $\exists v\, \phi$, we assume that v is an object which has the property expressed by ϕ and deduce the desired conclusion without making any further assumption about v.

So the formal rule we are looking for needs to express the fact that if we can derive a formula ψ, in which v has no free occurrences, from the assumption ϕ, and possibly also from other assumptions none of which contain any free occurrences of v, then we can also derive ψ from the assumption $\exists v\, \phi$ and the other assumptions. Thus we are led to the following rule.

Definition 3.2 Existential Hypothesis Rule (EH)

Let ψ be a formula which contains no free occurrences of the variable v. If on a line of a formal proof we have derived ψ from the assumption ϕ, which may contain free occurrences of v, and possibly from other assumptions, none of which contains free occurrences of v, then on any subsequent line we may introduce the formula ψ, which will depend on $\exists v\, \phi$ and the other assumptions, if any.

With the specified restriction on free occurrences of v, the rule can be shown to be logically valid. It is easy to see that the rule is machine-checkable.

Example 3.4

We show that

$$\exists y\, x = (y' + y') \vdash \exists z\, x = (z + z)$$

1	(1)	$\exists y\, x = (y' + y')$	Ass
2	(2)	$x = (y' + y')$	Ass
2	(3)	$\exists z\, x = (z + z)$	EI, 2
1	(4)	$\exists z\, x = (z + z)$	EH, 3

Because we are trying to derive $\exists z\, x = (z + z)$ from the assumption $\exists y\, x = (y' + y')$, we write down this latter formula as an assumption on line 1. The EH Rule tells us that one way to derive $\exists z\, x = (z + z)$ from this assumption is to derive it from $x = (y' + y')$ and then use the EH Rule. So we write down this formula as an assumption on line 2. We now think about how we could derive the desired conclusion $\exists z\, x = (z + z)$. The EI Rule is an appropriate rule to use if we want to derive a formula that begins with an existential quantifier. We note that if ψ is the formula $x = (z + z)$, then $x = (y' + y')$ is the formula $\psi(y'/z)$. The EI Rule entitles us to derive $\exists z\, \psi$ from $\psi(y'/z)$, and so, by using this rule on line 3, we can write the formula $\exists z\, x = (z + z)$ depending on the same assumptions as line 2. The variable y does not occur freely in $\exists z\, x = (z + z)$ and hence it is legitimate to use the EH Rule. Applying this to line 3, the effect is to replace the assumption $x = (y' + y')$ by the assumption $\exists y\, x = (y' + y')$. In this case there are no other assumptions to worry about. Hence we obtain line 4, which is what we have been aiming at.

◆

Notice that in this example the formula which we have derived on line 4 is exactly the same as the formula on line 3. The difference is in the assumptions involved. Line 3 depends on the formula on line 2, but line 4 has the formula on line 1 as its assumption. This is typical of the use of the Existential Hypothesis Rule. When we use it, its effect is only to change the assumption formulas which are in force. We said it was more subtle than the other rules!

Notice also that, typically, in preparation for a use of this rule, we introduced both the formulas $\exists y\, x = (y' + y')$ and $x = (y' + y')$ as assumptions. We didn't make the corresponding move in the informal proof of the theorem with which we began this section, but we could have done this by beginning the proof 'Assume that the matrix **A** has an inverse and suppose that **B** is the inverse of **A**'. Some people might say that this is stylistically preferably to our proof.

Example 3.5

We give a schematic proof to show that, for all formulas ϕ and θ,

$$\exists v\, \phi, \forall v\, (\phi \to \theta) \vdash \exists v\, \theta$$

Our standard first move will be to write down as assumptions the formulas $\exists v\, \phi$, $\forall v\, (\phi \to \theta)$, as we are aiming to show that $\exists v\, \theta$ can be derived from these assumptions. The EH Rule tells us that we will achieve this if we can derive $\exists v\, \theta$ from the assumptions ϕ, $\forall v\, (\phi \to \theta)$. So we next introduce ϕ as an assumption. Another standard technique when we have an assumption, as here, of the form $\forall v\, (\phi \to \theta)$ is to use the UE Rule to drop the initial universal quantifier. Once we do this, we will have derived both ϕ and $(\phi \to \theta)$. The Tautology Rule then enables us to derive θ, and we can then obtain the desired formula $\exists v\, \theta$ by use of the EI Rule. Let us follow this strategy through.

1	(1)	$\exists v\, \phi$	Ass
2	(2)	$\forall v\, (\phi \to \theta)$	Ass
3	(3)	ϕ	Ass
2	(4)	$(\phi \to \theta)$	UE, 2
2, 3	(5)	θ	Taut, 3, 4
2, 3	(6)	$\exists v\, \theta$	EI, 5
1, 2	(7)	$\exists v\, \theta$	EH, 6

On line 6 we have derived the formula $\exists v\, \theta$ we are aiming at, but not from the required assumptions. So we use the EH Rule to change the assumptions, that is, the assumption ϕ is replaced by the assumption $\exists v\, \phi$. Is this a correct use of the EH Rule? We need to check that the variable v has no free occurrences in the conclusion, $\exists v\, \theta$, nor in any of the assumptions, other than ϕ, on which this conclusion depends. Clearly, v has no free occurrences in $\exists v\, \theta$, as this formula begins with the quantifier $\exists v$. Likewise, v has no free occurrences in the only assumption, other than ϕ, on which $\exists v\, \theta$ depends on line (6), as this other assumption is $\forall v\, (\phi \to \theta)$. So our use of the EH Rule on line 7 is legitimate.

◆

Problem 3.2

Show that

$$\exists x\, (y' \cdot y') = x' \vdash \exists x\, (y' \cdot y') = x$$

Problem 3.3

Show that, for all formulas ϕ and ψ,

$$\exists v\, (\phi \,\&\, \psi) \vdash (\exists v\, \phi \,\&\, \exists v\, \psi)$$

Hint: Follow the strategy of Example 3.5 to derive $\exists v\, \phi$ and $\exists v\, \psi$ separately from $(\phi \,\&\, \psi)$. Then use the Tautology Rule and the EH Rule to complete the proof.

We have explained why the Existential Hypothesis Rule contains the following two requirements:

(a) the variable v has no free occurrences in the formula ψ;

(b) the variable v has no free occurrences in any assumption, other than ϕ, on which ψ depends.

It is instructive to give examples of what can go wrong when either of these requirements fails to hold. We emphasize that the following are *not* correct formal proofs.

(a)

1	(1)	$\exists v\, v = \mathbf{0}'$	Ass
2	(2)	$v = \mathbf{0}'$	Ass
1	(3)	$v = \mathbf{0}'$	EH, 2
1	(4)	$\forall v\, v = \mathbf{0}'$	UI, 3

Warning! This is not a correct proof!

(b)

1	(1)	$\exists v\, v = \mathbf{0}$	Ass
2	(2)	$\exists v\, v = \mathbf{0}'$	Ass
3	(3)	$v = \mathbf{0}$	Ass
4	(4)	$v = \mathbf{0}'$	Ass
3, 4	(5)	$(v = \mathbf{0} \,\&\, v = \mathbf{0}')$	Taut, 3, 4
3, 4	(6)	$\exists v\, (v = \mathbf{0} \,\&\, v = \mathbf{0}')$	EI, 5
2, 3	(7)	$\exists v\, (v = \mathbf{0} \,\&\, v = \mathbf{0}')$	EH, 6
1, 2	(8)	$\exists v\, (v = \mathbf{0} \,\&\, v = \mathbf{0}')$	EH, 7

Warning! This is not a correct proof!

The formula $\exists v\, v = \mathbf{0}'$ is true in the standard interpretation $\mathscr{N}$, but $\forall v\, v = \mathbf{0}'$ is false in this interpretation. So the formula $\forall v\, v = \mathbf{0}'$ is *not* a logical consequence of $\exists v\, v = \mathbf{0}'$. Hence, example (a), which purports to be a formal proof of $\forall v\, v = \mathbf{0}'$ from the assumption $\exists v\, v = \mathbf{0}'$, must have an error in it. Similarly, example (b) purports to be a formal proof of $\exists v\, (v = \mathbf{0} \,\&\, v = \mathbf{0}')$ from the assumptions $\exists v\, v = \mathbf{0}$ and $\exists v\, v = \mathbf{0}'$. These assumptions are true in $\mathscr{N}$ but the conclusion is false. Thus $\exists v\, (v = \mathbf{0} \,\&\, v = \mathbf{0}')$ is *not* a logical consequence of the formulas $\exists v\, v = \mathbf{0}$, $\exists v\, v = \mathbf{0}'$. It follows that there must be an error in the purported proof.

Problem 3.4

Find the errors in each of the above examples. If the Existential Hypothesis Rule has been used inappropriately, specify which requirement of this rule fails to hold.

In contrast, the following example is a schematic proof in which the Existential Hypothesis Rule is used correctly.

Example 3.6

We show that, for all formulas ϕ,

$$\exists u\, \exists v\, \phi \vdash \exists v\, \exists u\, \phi$$

If we can derive $\exists v\, \exists u\, \phi$ from the assumption $\exists v\, \phi$, then we can use the EH Rule to derive it from the assumption $\exists u\, \exists v\, \phi$. Likewise, we can use the EH Rule to derive $\exists v\, \exists u\, \phi$ from the assumption $\exists v\, \phi$ if we can derive it from the assumption ϕ. The EI Rule enables us to derive $\exists v\, \exists u\, \phi$ from ϕ. Thus we arrive at the following schematic proof.

1	(1)	$\exists u\, \exists v\, \phi$	Ass
2	(2)	$\exists v\, \phi$	Ass
3	(3)	ϕ	Ass
3	(4)	$\exists u\, \phi$	EI, 3
3	(5)	$\exists v\, \exists u\, \phi$	EI, 4
2	(6)	$\exists v\, \exists u\, \phi$	EH, 5
1	(7)	$\exists v\, \exists u\, \phi$	EH, 6

Since neither of the variables u and v can have any free occurrences in the formula $\exists v\, \exists u\, \phi$, our uses of the EH Rule on lines 6 and 7 are correct. ♦

Problem 3.5

Show that, for all formulas ϕ and ψ,

$$\exists v\, (\phi \vee \psi), \forall v\, \neg\phi \vdash \exists v\, \psi$$

Here is an example of a schematic proof which uses each quantifier rule once.

Example 3.7

For each formula ϕ,

$$\exists u\, \forall v\, \phi \vdash \forall v\, \exists u\, \phi$$

1	(1)	$\exists u\, \forall v\, \phi$	Ass
2	(2)	$\forall v\, \phi$	Ass
2	(3)	ϕ	UE, 2
2	(4)	$\exists u\, \phi$	EI, 3
2	(5)	$\forall v\, \exists u\, \phi$	UI, 4
1	(6)	$\forall v\, \exists u\, \phi$	EH, 5

You should check that this is a correct schematic proof. In particular check that, when we have used the UI Rule on line 5 and the EH Rule on line 6, the requirements of these rules in relation to free occurrences of variables are satisfied. ♦

You have now met and used seven of the nine rules of proof. You may have found it difficult to see how to use some of these rules, particularly the Tautology Rule, in constructing formal proofs. To help you with this, the problem of finding formal proofs is discussed in *Unit 6*, where you will also be introduced to the final two rules of proof.

SUMMARY

We began by giving a precise account of what we mean by a *formal proof.* This was based on a look at what informal proofs in everyday mathematics are like. We said that a formal proof must be both logically valid and checkable by a machine.

We specified that a formal proof should consist of a finite sequence of formulas, and that each step must be justified by one of our *rules of proof.* This led us to set out a formal proof in lines, with one formula on each line. The lines are numbered for ease of reference, and each line is annotated on the right to indicate the rule of proof which has been used and, where appropriate, the earlier line or lines to which the rule has been applied.

We saw that both informal and formal proofs involve drawing conclusions from assumptions and that it is important to keep track of which assumptions each line of a proof depends on. Each rule of proof includes a specification of which assumptions the conclusion depends on. We annotate proofs by listing, on the left of each line number, the numbers of the formulas which are used as assumptions in deriving that line.

Having described the general structure of a formal proof, we began to describe the rules of proof that we use.

The Assumption Rule is used to introduce assumptions into formal proofs. The main rule for handling arguments involving the connectives is the powerful Tautology Rule. When it comes to finding formal proofs, the use of this rule depends on being able to spot appropriate tautologies. The Conditional Proof Rule is useful when the formula we are trying to derive is an implication.

We came next to the rules for handling the quantifiers. We needed to make an important distinction between *free* and *bound* occurrences of variables in formulas. We saw that, from the formation rules described in *Unit 4*, it is not difficult to obtain an algorithm for deciding which occurrences of variables in formulas are free and which are bound.

Our rules for the quantifiers all involve restrictions which depend on whether certain variables have free occurrences in the formulas we are using. We saw that without these restrictions the rules would not be logically valid. We have two rules for each quantifier. For the universal quantifier we have both an Introduction and an Elimination Rule. We also have an Introduction Rule for the existential quantifier, but the second rule for this quantifier, the Existential Hypothesis Rule is rather more subtle. When we apply this rule to a particular line of a formal proof, the formula on the line is unchanged, but one of the assumptions on which it depends is changed by the prefixing of an existential quantifier.

We have not yet completed listing all the rules of proof of our formal system. We do this in the next unit where we introduce two rules dealing with the identity symbol. We also address the practical problem of finding formal proofs, and offer some guidelines to help with this task. *Unit 6* ends with a discussion of how we augment the formal system by adding axioms to describe properties of natural numbers, which is the ultimate focus of our attention.

OBJECTIVES

We list those topics on which we may set assessment questions to test your understanding of this unit.

After working through the unit you should be able to:

(a) understand the meaning and use of each of the rules Ass, Taut and CP;

(b) determine whether the occurrences of variables in formulas are free or bound;

(c) determine whether or not a given term may be freely substituted for a given variable in a given formula;

(d) understand the meaning and use of each of the rules UE, UI, EI and EH and why there are restrictions on the variables and terms they involve;

(e) check a purported formal proof to see whether the uses of the seven rules Ass, Taut, CP, UE, UI, EI and EH are legitimate;

(f) add the correct assumption numbers to the lines of a formal proof from which they are missing;

(g) construct simple formal proofs using the rules listed in (e).

ADDITIONAL EXERCISES

Most of these exercises provide further practice, should you feel you need it, in handling the main ideas in the unit on which you are likely to be assessed.

There are a few harder problems, labelled as such in the margin. These are harder than any of the problems you are likely to encounter in the assessment and are included solely as challenges for the interested student.

Section 1

1 The following is a correct schematic proof from which the assumption numbers have been omitted.

(1)	$\neg\phi$	Ass
(2)	$\neg\psi$	Ass
(3)	$(\neg\phi \,\&\, \neg\psi)$	Taut, 1, 2
(4)	$(\phi \vee \psi)$	Ass
(5)	$\neg(\neg\phi \,\&\, \neg\psi)$	Taut, 4
(6)	$((\neg\phi \,\&\, \neg\psi) \,\&\, \neg(\neg\phi \,\&\, \neg\psi))$	Taut, 3, 5
(7)	$(\neg\psi \rightarrow ((\neg\phi \,\&\, \neg\psi) \,\&\, \neg(\neg\phi \,\&\, \neg\psi)))$	CP, 6
(8)	ψ	Taut, 7
(9)	$((\phi \vee \psi) \rightarrow \psi)$	CP, 8

(a) What are the assumptions in force on each line?

(b) What tautologies are being used on lines 3, 5 and 8?

(c) Show that the formulas used to obtain lines 5 and 8 are indeed tautologies.

2 Find all the mistakes in the following 'schematic proof'; when investigating a particular line, find the errors on that line, if any, under the assumption that all the previous lines are correct.

Warning! This is not a correct proof!

1	(1)	$(\phi \vee \psi)$	Taut
1	(2)	$(\phi \,\&\, \psi)$	Taut, 1
2	(3)	ϕ	Taut, 2
1	(4)	ψ	Taut, 1, 2
	(5)	$(\psi \rightarrow (\phi \vee \psi))$	CP, 4
1	(6)	$(\phi \vee \psi)$	Taut, 4, 5

3 Determine a schematic proof of

$$((\phi \rightarrow \psi) \rightarrow (\neg\psi \rightarrow \neg\phi))$$

which depends on no assumptions.

4 Prove that if ϕ is a formula which is a tautology, then there is a formal proof of ϕ which depends on no assumptions. *Hints*: Suppose ϕ is a tautology. Knowing nothing else about ϕ, the only hope of deriving it would seem to be a use of the Tautology Rule. This means finding a formula θ such that $(\theta \rightarrow \phi)$ is a tautology, where we have previously derived θ. The Tautology Rule could then be used to derive ϕ depending on the same assumptions as θ. Since we want to derive ϕ from no assumptions at all, we need to find a formula θ that can be derived from no assumptions. Can you find such a θ for which $(\theta \rightarrow \phi)$ is a tautology?

Harder problem

Section 2

1 For each of the following formulas, determine which occurrences of variables are free and which are bound.

(a) $\forall y\,(\exists t\,(y+x)=t \rightarrow x'=(t\cdot z))$

(b) $\forall t\,\exists x\,(\forall y\,x=y \rightarrow \exists z\,z'=(t\cdot x))$

(c) $(\forall x\,\exists t\,x'=t' \vee \exists y\,(x+y)=t)$

2 For each of the formulas in the preceding question, write down the result of substituting the term $(x\cdot t)$ for each free occurrence of x in the formula. In which of the formulas may this term be freely substituted for x?

3 Show that

$$\forall x\,\forall y\,(x+y)=(y+x) \vdash \forall x\,(x+\mathbf{0})=(\mathbf{0}+x)$$

4 Show that, for all formulas ϕ and ψ,

$$\forall v\,(\phi \rightarrow \psi) \vdash (\phi \rightarrow \forall v\,\psi)$$

provided that the variable v does not occur freely in ϕ. For which step(s) of your argument do you need the condition that v does not occur freely in ϕ?

5 Show that, for all formulas ϕ, ψ and θ,

$$\forall v\,(\phi \rightarrow \psi), \forall v\,(\psi \rightarrow \theta) \vdash \forall v\,(\phi \rightarrow \theta)$$

Section 3

1 Show that, for all formulas ϕ and ψ,

$$\forall v\,(\phi \rightarrow \psi) \vdash (\exists v\,\phi \rightarrow \psi)$$

provided that the variable v does not occur freely in ψ. For which step(s) of your argument do you need the condition that v does not occur freely in ψ?

2 The following correct schematic proof is incomplete because the assumption numbers are missing. Fill in these numbers.

(1)	$\forall v\,(\theta \rightarrow \psi)$	Ass
(2)	$(\theta \rightarrow \psi)$	UE, 1
(3)	$\exists v\,\theta$	Ass
(4)	$(\theta \rightarrow (\theta \rightarrow \psi))$	Taut, 2
(5)	θ	Ass
(6)	$(\theta \rightarrow \psi)$	Taut, 4, 5
(7)	$\exists v\,(\theta \rightarrow \psi)$	EI, 6
(8)	$\exists v\,(\theta \rightarrow \psi)$	EH, 7
(9)	$(\exists v\,\theta \rightarrow \exists v\,(\theta \rightarrow \psi))$	CP, 8

3 Find the mistake(s) in the following 'schematic proof' that, for all formulas ϕ and ψ,

$$\exists v\, \phi, \exists v\, \psi \vdash \exists v\, (\phi \,\&\, \psi)$$

Warning! This is not a correct proof!

1	(1)	$\exists v\, \phi$	Ass
2	(2)	$\exists v\, \psi$	Ass
3	(3)	ϕ	Ass
4	(4)	ψ	Ass
3, 4	(5)	$(\phi \,\&\, \psi)$	Taut, 3, 4
3, 4	(6)	$\exists v\, (\phi \,\&\, \psi)$	EI, 5
1, 4	(7)	$\exists v\, (\phi \,\&\, \psi)$	EH, 6
1, 2	(8)	$\exists v\, (\phi \,\&\, \psi)$	EH, 7

4 Show that

$$\exists x\, x = \mathbf{0} \vdash \exists y\, y = \mathbf{0}$$

5 The Existential Hypothesis Rule is stated as follows.

> Let ψ be a formula which contains no free occurrences of the variable v. If on a line of a formal proof we have derived ψ from the assumption ϕ, which may contain free occurrences of v, and possibly from other assumptions, none of which contains free occurrences of v, then on any subsequent line we may introduce the formula ψ, which will depend on $\exists v\, \phi$ and the other assumptions, if any.

Give an example to show that the rule would not be logically valid if the condition that ψ contains no free occurrences of v is omitted. This means finding a formula ψ containing free occurrences of v, a formula ϕ which may contain free occurrences of v and possibly formulas $\phi_1, \phi_2, \ldots, \phi_k$ in which v does not occur freely such that ψ is a logical consequence of ϕ and $\phi_1, \phi_2, \ldots, \phi_k$, but with ψ not a logical consequence of $\exists v\, \phi$ and $\phi_1, \phi_2, \ldots, \phi_k$. *Hint*: The discussion following Problem 3.3 should help.

6 The Existential Quantifier Introduction Rule is stated as follows.

Harder problem

> If τ is a term which may be freely substituted for the variable v in the formula ϕ, and the formula $\phi(\tau/v)$ occurs on a line of a formal proof, then on any subsequent line we may introduce the formula $\exists v\, \phi$, which will depend on the same assumptions as does $\phi(\tau/v)$.

Give an example to show that the rule would not be logically valid if the condition that the term τ may be freely substituted for the variable v in the formula ϕ is omitted. This means finding a formula ϕ, a term τ not freely substitutable for v in ϕ and possibly formulas $\phi_1, \phi_2, \ldots, \phi_k$ such that $\phi(\tau/v)$, but not $\exists v\, \phi$, is a logical consequence of $\phi_1, \phi_2, \ldots, \phi_k$.

SOLUTIONS TO THE PROBLEMS

Solution 1.1

(a) (i) The schematic proof with the assumption numbers added is as follows.

1	(1)	ψ	Ass
2	(2)	ϕ	Ass
1	(3)	$(\phi \vee \psi)$	Taut, 1
1, 2	(4)	$(\phi \,\&\, \psi)$	Taut, 1, 2
1, 2	(5)	$((\phi \vee \psi) \,\&\, (\phi \,\&\, \psi))$	Taut, 3, 4
2	(6)	$(\psi \rightarrow ((\phi \vee \psi) \,\&\, (\phi \,\&\, \psi)))$	CP, 5

(ii) The tautologies which have been used are as follows.

line 3: $(\psi \rightarrow (\phi \vee \psi))$
line 4: $((\psi \,\&\, \phi) \rightarrow (\phi \,\&\, \psi))$
line 5: $((((\phi \vee \psi) \,\&\, (\phi \,\&\, \psi)) \rightarrow ((\phi \vee \psi) \,\&\, (\phi \,\&\, \psi)))$

(b) (i) The schematic proof with the assumption numbers added is as follows.

1	(1)	$(\phi \vee \chi)$	Ass
2	(2)	$\neg\chi$	Ass
1, 2	(3)	ϕ	Taut, 1, 2
4	(4)	ψ	Ass
1, 2, 4	(5)	$(\phi \,\&\, \psi)$	Taut, 3, 4
1, 4	(6)	$(\neg\chi \rightarrow (\phi \,\&\, \psi))$	CP, 5
1, 4	(7)	$(\chi \vee (\phi \,\&\, \psi))$	Taut, 6
1	(8)	$(\psi \rightarrow (\chi \vee (\phi \,\&\, \psi)))$	CP, 7

(ii) The tautologies which have been used are as follows.

line 3: $(((\phi \vee \chi) \,\&\, \neg\chi) \rightarrow \phi)$
line 5: $((\phi \,\&\, \psi) \rightarrow (\phi \,\&\, \psi))$
line 7: $((\neg\chi \rightarrow (\phi \,\&\, \psi)) \rightarrow (\chi \vee (\phi \,\&\, \psi)))$

Solution 1.2

(a) One possible schematic proof is as follows.

1	(1)	ϕ	Ass
2	(2)	$\neg\psi$	Ass
1, 2	(3)	$\neg(\phi \rightarrow \psi)$	Taut, 1, 2

(b) All that we need to do is to append to the above schematic proof two lines on which we use the Conditional Proof Rule, as follows.

1	(4)	$(\neg\psi \rightarrow \neg(\phi \rightarrow \psi))$	CP, 3
	(5)	$(\phi \rightarrow (\neg\psi \rightarrow \neg(\phi \rightarrow \psi)))$	CP, 4

Solution 1.3

(a)

1	(1)	ϕ	Ass
2	(2)	$(\phi \to \psi)$	Ass
1, 2	(3)	ψ	Taut, 1, 2
1	(4)	$((\phi \to \psi) \to \psi)$	CP, 3
	(5)	$(\phi \to ((\phi \to \psi) \to \psi))$	CP, 4

(b)

1	(1)	$(\neg\phi \to \neg\psi)$	Ass
2	(2)	$(\neg\phi \to \psi)$	Ass
1, 2	(3)	ϕ	Taut, 1, 2
1	(4)	$((\neg\phi \to \psi) \to \phi)$	CP, 3
	(5)	$((\neg\phi \to \neg\psi) \to ((\neg\phi \to \psi) \to \phi))$	CP, 4

The same strategy has been used in both parts (a) and (b).

In part (a) we are aiming to show that we can derive the formula $(\phi \to ((\phi \to \psi) \to \psi))$ depending on no assumptions. Since this formula is an implication, a use of the Conditional Proof Rule suggests itself: if we can derive $((\phi \to \psi) \to \psi)$ depending on the assumption ϕ, then using Conditional Proof Rule it follows that $(\phi \to ((\phi \to \psi) \to \psi))$ can be derived depending on no assumptions. Since $((\phi \to \psi) \to \psi)$ is also an implication, another use of the Conditional Proof Rule indicates itself: if we show that the formula ψ can be derived from the assumptions ϕ and $(\phi \to \psi)$, then by Conditional Proof Rule $((\phi \to \psi) \to \psi)$ can be derived from the assumption ϕ. We can derive ψ from the assumptions ϕ and $(\phi \to \psi)$ in one step using the Tautology Rule provided that the formula

$$((\phi \,\&\, (\phi \to \psi)) \to \psi)$$

is a tautology. Using a truth table, it is easy to check that this is so.

Essentially the same strategy is used in part (b). The use of the Tautology Rule on line 3 is correct provided that the formula

$$(((\neg\phi \to \neg\psi) \,\&\, (\neg\phi \to \psi)) \to \phi)$$

is a tautology. Again, using a truth table, it is easily checked that this is so.

Solution 2.1

In each case we have underlined the bound occurrences of variables. The given formula is a sentence if all the variables in it are underlined, that is, if all the occurrences of variables are bound occurrences.

(a) $\forall \underline{x}\, \exists \underline{y} \neg\, \underline{x} = \underline{y}$. This is a sentence.

(b) $((x = y \lor \exists \underline{t}\, (x + \underline{t}) = y) \lor \exists \underline{t}\, (y + \underline{t}) = x)$. This is not a sentence.

(c) $(\exists \underline{z}\, (\underline{z} \cdot x) = y \to \forall \underline{y}\, (\exists \underline{z}\, (\underline{z} \cdot \underline{y}) = x \to \neg \exists \underline{x}\, (\underline{x} \cdot \underline{x}) = \underline{y}))$. This is not a sentence.

(d) $\exists \underline{x}\, (\underline{x} = y \lor \neg\, \underline{x} = y)$. This is not a sentence.

(e) $(\exists \underline{x}\, \underline{x} = y \lor \neg\, x = y)$. This is not a sentence.

Notice the effect of the different arrangement of the brackets in (d) and (e).

Solution 2.2

(a) $\exists y\,(y \cdot y) = ((z'' + \mathbf{0}) \cdot ((z'' + \mathbf{0}) + y))$

(b) Here only the final occurrence of x is a free occurrence, and hence we only replace this occurrence of x by $\mathbf{0}'$. Thus we obtain

$$(\exists x\,(x \cdot x) = y \,\&\, \mathbf{0}' = (z + y))$$

(c) Here there are no free occurrences of x and hence $\phi(\tau/v)$ is the same as ϕ, that is

$$\exists x\,((x \cdot x) = y \,\&\, x = (z + y))$$

(d) $\exists x\,((x \cdot x) = \mathbf{0}' \,\&\, x = (z + \mathbf{0}'))$

(e) $\exists y\, y = (y' + y')$

(f) $\exists x\,(x + x) = (x \cdot x)$

(g) Since τ is the same as v, $\phi(\tau/v)$ is the same as ϕ, that is

$$\exists x\,(z + x) = (z \cdot x)$$

Solution 2.3

(a) The term $(z'' \cdot y)$ may be freely substituted for x in the given formula ϕ. Here $\phi(\tau/v)$ is

$$(\exists z\, z = y' \,\&\, \exists t\,(t \cdot t) = (z'' \cdot y))$$

and the occurrences of the variables y and z in τ give rise to free occurrences in $\phi(\tau/v)$.

(b) The term $(z'' \cdot y)$ may *not* be freely substituted for x in ϕ. Here $\phi(\tau/v)$ is

$$\exists \underline{z}\,(\underline{z} = y' \,\&\, \exists \underline{t}\,(\underline{t} \cdot \underline{t}) = \underline{z}'' \cdot y))$$

where the bound occurrences of variables have been underlined. The variable z in τ becomes bound in $\phi(\tau/v)$.

(c) The term $(y + x)$ may be freely substituted for y in ϕ. Here $\phi(\tau/v)$ is

$$(\forall x\, \exists z\, x = (z + z) \rightarrow \exists t\,(t + t) = ((y + x) + z))$$

and the occurrences of x and y in τ give rise to free occurrences of these variables in $\phi(\tau/v)$.

(d) There are no free occurrences of y in ϕ, so $\phi(\tau/v)$ is identical to ϕ and, in a vacuous way, none of the variables in τ becomes bound when we replace the free occurrences of y in ϕ by τ. Thus τ may be freely substituted for y in ϕ.

(e) The term τ can be freely substituted for x in ϕ. Since τ is the same as x, $\phi(\tau/v)$ is the same as ϕ and no problem arises.

Solution 2.4

1	(1)	$\forall x \forall y\,(x + y) = (y + x)$	Ass
1	(2)	$\forall y\,(\mathbf{0}' + y) = (y + \mathbf{0}')$	UE, 1
1	(3)	$(\mathbf{0}' + \mathbf{0}'') = (\mathbf{0}'' + \mathbf{0}')$	UE, 2

Solution 2.5

The formula $\forall x\, \exists y \neg x = y$ can be regarded as being $\forall x\, \phi$, where ϕ is the formula $\exists y \neg x = y$. The formula $\exists y \neg \tau = y$ is then $\phi(\tau/x)$.

To see that the answer to the question posed is 'no', we need to find a term τ that is not freely substitutable for x in ϕ. The simplest example is to take τ to be the variable y. Thanks to the $\exists y$, this term cannot be freely substituted for x in $\exists y \neg x = y$.

With this choice for τ, we now need to show that $\exists y \neg y = y$ is not a logical consequence of $\forall x\, \exists y \neg x = y$. This means giving an interpretation in which $\forall x\, \exists y \neg x = y$ is true and $\exists y \neg y = y$ is false. We shall take the standard interpretation $\mathscr{N}$ with domain $\mathbb{N}$. In this interpretation $\forall x\, \exists y \neg x = y$ is true as for any natural number x there is always a number y to which it is not equal, while $\exists y \neg y = y$ is false as there is no number not equal to itself.

Any interpretation with a domain containing at least two elements will do.

Solution 2.6

1	(1)	$\forall x\, \forall y\, \forall z\, ((x + y) \cdot z) = ((x \cdot z) + (y \cdot z))$	Ass
1	(2)	$\forall y\, \forall z\, ((x + y) \cdot z) = ((x \cdot z) + (y \cdot z))$	UE, 1
1	(3)	$\forall z\, ((x + x) \cdot z) = ((x \cdot z) + (x \cdot z))$	UE, 2
1	(4)	$((x + x) \cdot x) = ((x \cdot x) + (x \cdot x))$	UE, 3
1	(5)	$\forall x\, ((x + x) \cdot x) = ((x \cdot x) + (x \cdot x))$	UI, 4

In each case where we use the UE Rule to replace a formula of the form $\forall v\, \phi$ by $\phi(\tau/v)$, the term τ is just the variable x itself.

Solution 2.7

We first derive $\forall v\, \phi$ and $\forall v\, \psi$ separately and then use the Tautology Rule to obtain $(\forall v\, \phi \,\&\, \forall v\, \psi)$. In deriving $\forall v\, \phi$, we first derive ϕ and then apply the UI Rule. We then use the same strategy to derive $\forall v\, \psi$.

1	(1)	$\forall v\, (\phi \,\&\, \psi)$	Ass
1	(2)	$(\phi \,\&\, \psi)$	UE, 1
1	(3)	ϕ	Taut, 2
1	(4)	$\forall v\, \phi$	UI, 3
1	(5)	ψ	Taut, 2
1	(6)	$\forall v\, \psi$	UI, 5
1	(7)	$(\forall v\, \phi \,\&\, \forall v\, \psi)$	Taut, 4, 6

Solution 2.8

Let ϕ be the formula $v = \mathbf{0}$, as suggested in the hint. Let ϕ_1 be the same formula, so v occurs freely in ϕ_1. Then trivially ϕ is a logical consequence of ϕ_1. But we can show that $\forall v\, \phi$ is not a logical consequence of ϕ_1. For instance, take the standard interpretation $\mathscr{N}$ and give v the value 0 from the domain $\mathbb{N}$. Then the formula $v = \mathbf{0}$ is true, but $\forall v\, v = \mathbf{0}$ is false, as it is not the case that all numbers v are equal to 0.

Any interpretation with a domain containing at least two elements will do.

Solution 3.1

(a) If we let ϕ be the formula $\forall x\,(x+y)=x$, then $\phi(\mathbf{0}/y)$ is $\forall x\,(x+\mathbf{0})=x$ and from $\phi(\mathbf{0}/y)$ we can derive $\exists y\,\phi$. So we have the following formal proof.

1	(1)	$\forall x\,(x+\mathbf{0})=x$	Ass
1	(2)	$\exists y\forall x\,(x+y)=x$	EI, 1

(b)

1	(1)	$\forall x\,(x\cdot\mathbf{0}')=x$	Ass
1	(2)	$(x\cdot\mathbf{0}')=x$	UE, 1
1	(3)	$\exists y\,(x\cdot y)=x$	EI, 2
1	(4)	$\forall x\exists y\,(x\cdot y)=x$	UI, 3

(c)

1	(1)	$(\mathbf{0}+\mathbf{0})=\mathbf{0}$	Ass
1	(2)	$\exists y\,(y+y)=\mathbf{0}$	EI, 1

(d)

1	(1)	$\forall x\,(x+\mathbf{0})=x$	Ass
1	(2)	$(\mathbf{0}+\mathbf{0})=\mathbf{0}$	UE, 1
1	(3)	$\exists x\,(x+x)=x$	EI, 2

If we let ϕ be the formula $(x+\mathbf{0})=x$ and ψ be the formula $(x+x)=x$, then the formula on line 1 is $\forall x\,\phi$ and we use the UE Rule to derive $\phi(\mathbf{0}/x)$ on line 2. But this formula is also $\psi(\mathbf{0}/x)$ and so we use the EI Rule to derive $\exists x\,\psi$ on line 3.

Solution 3.2

1	(1)	$\exists x\,(y'\cdot y')=x'$	Ass
2	(2)	$(y'\cdot y')=x'$	Ass
2	(3)	$\exists x\,(y'\cdot y')=x$	EI, 2
1	(4)	$\exists x\,(y'\cdot y')=x$	EH, 3

If ϕ is the formula $(y'\cdot y')=x$, then the formula $(y'\cdot y')=x'$ which appears on line 2 is $\phi(x'/x)$. Hence the use of the EI Rule to derive $\exists x\,\phi$ is legitimate. Since the variable x does not occur freely in $\exists x\,\phi$, the use of the EH rule on line 4 is also legitimate.

Solution 3.3

1	(1)	$\exists v\,(\phi\,\&\,\psi)$	Ass
2	(2)	$(\phi\,\&\,\psi)$	Ass
2	(3)	ϕ	Taut, 2
2	(4)	$\exists v\,\phi$	EI, 3
2	(5)	ψ	Taut, 2
2	(6)	$\exists v\,\psi$	EI, 5
2	(7)	$(\exists v\,\phi\,\&\,\exists v\,\psi)$	Taut, 4, 6
1	(8)	$(\exists v\,\phi\,\&\,\exists v\,\psi)$	EH, 7

Solution 3.4

(a) The error occurs on line 3. The use of the EH Rule on this line is not valid because the variable v has a free occurrence in the formula on line 2 to which the rule has been applied. Thus, in the terms of the discussion preceding the problem, requirement (a) of the EH rule does not hold.

(b) The error occurs on line 7. The use of the EH Rule on this line is not correct because the variable v occurs freely in the formula $v=\mathbf{0}$, that is, the formula on line 3, which is used as an assumption on both lines 6 and 7. So requirement (b) does not hold.

Note that the use of the EH Rule on line 8 *is* valid as the variable v does not occur freely in the formula on line 2, that is $\exists v\,v=\mathbf{0}'$, which is used as an assumption on both lines 7 and 8. However, a single illegitimate use of a rule is enough to invalidate the entire formal proof.

Solution 3.5

1	(1)	$\exists v\,(\phi \vee \psi)$	Ass
2	(2)	$\forall v\,\neg\phi$	Ass
3	(3)	$(\phi \vee \psi)$	Ass
2	(4)	$\neg\phi$	UE, 2
2, 3	(5)	ψ	Taut, 3, 4
2, 3	(6)	$\exists v\,\psi$	EI, 5
1, 2	(7)	$\exists v\,\psi$	EH, 6

The strategy used to find this schematic proof follows ideas used in earlier examples. We begin by writing down as assumptions the two formulas from which we are aiming to derive $\exists v\,\psi$. Experience with the EH Rule tells us that to derive a formula from the assumption $\exists v\,(\phi \vee \psi)$ it would be a good move to derive it from $(\phi \vee \psi)$. So we introduce this formula as an assumption on line 3. Experience also tells us that with an assumption such as $\forall v\,\neg\phi$ it would probably be helpful to use the UE Rule to drop the universal quantifier, and we have done this on line 4.

We now turn our attention to our ultimate aim of deriving the formula $\exists v\,\psi$. We know that if we can derive ψ then we can obtain $\exists v\,\psi$ by using the EI Rule. So we look to see if we can derive ψ from the formulas $(\phi \vee \psi)$ and $\neg\phi$ on lines 3 and 4. We can achieve this by a single use of the Tautology Rule provided that the formula $(((\phi \vee \psi)\,\&\,\neg\phi) \rightarrow \psi)$ is a tautology. A truth table shows that it is a tautology and this gives us line 5. Using what should by now be familiar applications of the EI and EH Rules, we then get the desired conclusion $\exists v\,\psi$ depending on the desired assumptions on line 7.

SOLUTIONS TO ADDITIONAL EXERCISES

Section 1

1 (a)

1	(1)	$\neg\phi$	Ass
2	(2)	$\neg\psi$	Ass
1, 2	(3)	$(\neg\psi\,\&\,\neg\psi)$	Taut, 1, 2
4	(4)	$(\phi \vee \psi)$	Ass
4	(5)	$\neg(\neg\phi\,\&\,\neg\psi)$	Taut, 4
1, 2, 4	(6)	$((\neg\phi\,\&\,\neg\psi)\,\&\,\neg(\neg\phi\,\&\,\neg\psi))$	Taut, 3, 5
1, 4	(7)	$(\neg\psi \rightarrow ((\neg\phi\,\&\,\neg\psi)\,\&\,\neg(\neg\psi\,\&\,\neg\psi)))$	CP, 6
1, 4	(8)	ψ	Taut, 7
1	(9)	$((\psi \vee \psi) \rightarrow \psi)$	CP, 8

(b) line 3: $((\neg\phi\,\&\,\neg\psi) \rightarrow (\neg\phi\,\&\,\neg\psi))$
line 5: $((\phi \vee \psi) \rightarrow \neg(\neg\phi\,\&\,\neg\psi))$
line 8: $((\neg\psi \rightarrow ((\neg\phi\,\&\,\neg\psi)\,\&\,\neg(\neg\phi\,\&\,\neg\psi))) \rightarrow \psi)$

(c) Truth tables must be provided.

(ϕ	$\vee$	ψ)	$\rightarrow$	$\neg$	($\neg$	ϕ	&	$\neg$	ψ))
1	1	1	1	1	0	1	0	0	1
1	1	0	1	1	0	1	0	1	0
0	1	1	1	1	1	0	0	0	1
0	0	0	1	0	1	0	1	1	0

↑
tautology .

(($\neg$	ψ	$\rightarrow$	(($\neg$	ϕ	&	$\neg$	ψ)	&	$\neg$	($\neg$	ϕ	&	$\neg$	ψ)))	$\rightarrow$	ψ)
0	1	1	0	1	0	0	1	0	1	0	1	0	0	1	1	1
1	0	0	0	1	0	1	0	0	1	0	1	0	1	0	1	0
0	1	1	1	0	0	0	1	0	1	1	0	0	0	1	1	1
1	0	0	1	0	1	1	0	0	0	1	0	1	1	0	1	0

↑
tautology

2 Line 1: With the rules introduced so far, the first line of a proof can only be obtained by an application of Assumption Rule; to correct the line, replace 'Taut' by 'Ass'.

Line 2: This is an incorrect use of the Tautology Rule, as $((\phi \vee \psi) \rightarrow (\phi \,\&\, \psi))$ is not a tautology.

Line 3: A tautology has been used but, for a correct use of the Tautology Rule, the assumptions in force on line 3 should be those in force on line 2, namely assumption 1.

Line 4: This line is correct.

Line 5: The wrong formula has been derived. The formula on line 4 should appear as the consequent of the implication (i.e. after the → symbol) on line 5, and the formula on line 1 should appear as the antecedent (i.e. before the → symbol). The correct formula derived by the Conditional Proof Rule is $((\phi \vee \psi) \rightarrow \psi)$.

Line 6: This line is correct.

3 There are many correct proofs. We give two examples.

From the pattern of what is to be proved, and in particular the position of some of the occurrences of →, we see that if we can prove $(\neg\psi \rightarrow \neg\phi)$ from the assumption $(\phi \rightarrow \psi)$, then a use of the Conditional Proof Rule will give a proof of $((\phi \rightarrow \psi) \rightarrow (\neg\psi \rightarrow \neg\phi))$, as follows.

1	(1)	$(\phi \rightarrow \psi)$	Ass
1	(2)	$(\neg\psi \rightarrow \neg\phi)$	Taut, 1
	(3)	$((\phi \rightarrow \psi) \rightarrow (\neg\psi \rightarrow \neg\phi))$	CP, 2

The tautology used to obtain line 2 is $((\phi \rightarrow \psi) \rightarrow (\neg\psi \rightarrow \neg\phi))$.

Equally, a proof of $\neg\phi$ from the assumptions $(\phi \rightarrow \psi)$ and $\neg\psi$ can be turned into a proof of $((\phi \rightarrow \psi) \rightarrow (\neg\psi \rightarrow \neg\phi))$ by two uses of the Conditional Proof Rule, as follows.

1	(1)	$(\phi \rightarrow \psi)$	Ass
2	(2)	$\neg\psi$	Ass
1, 2	(3)	$\neg\phi$	Taut, 1, 2
1	(4)	$(\neg\psi \rightarrow \neg\phi)$	CP, 3
	(5)	$((\phi \rightarrow \psi) \rightarrow (\neg\psi \rightarrow \neg\phi))$	CP, 4

The tautology used to obtain line 3 is $(((\phi \rightarrow \psi) \,\&\, \neg\psi) \rightarrow \neg\phi)$.

4 We note first that if ϕ is a tautology then, for each formula θ, the formula $(\theta \to \phi)$ is also a tautology. For suppose ϕ is a tautology. Then, on each row of the truth table for $(\theta \to \phi)$, ϕ gets the value 1. We see from the truth table for $\to$ that it follows that $(\theta \to \phi)$ also gets the value 1 on each row, and hence it is a tautology.

It follows that, if ϕ is a tautology, we can use the Tautology Rule to derive ϕ from θ, whichever formula we choose for θ. If θ can be derived using no assumptions, then in this way we get a derivation of ϕ using no assumptions. Thus we can use for θ any formula which can be derived using no assumptions.

The following is a simple schematic proof that makes use of this idea.

1	(1)	ϕ	Ass
	(2)	$(\phi \to \phi)$	CP, 1
	(3)	ϕ	Taut, 2

The formula $(\phi \to \phi)$ plays the role of θ in the above discussion.

Section 2

1 (a) $\forall \underline{y}\,(\exists \underline{t}\,(\underline{y} + x) = \underline{t} \to x' = (t \cdot z))$

The bound occurrences of variables are underlined.

(b) All the occurrences of variables are bound. Thus the formula is a sentence.

(c) $(\forall \underline{x}\, \exists \underline{t}\, \underline{x}' = \underline{t}' \lor \exists \underline{y}\,(x + \underline{y}) = t)$

Again we have underlined the bound occurrences of the variables.

2 (a) The formula that results from substituting the term $(x \cdot t)$ for the free occurrences of x is

$$\forall y\,(\exists t\,(y + (x \cdot t)) = t \to (x \cdot t)' = (t \cdot z))$$

↑

bound

The term may not be freely substituted for x as a new bound variable is introduced.

(b) As there are no free occurrences of x, the formula is unchanged. Thus the term may be freely substituted.

See the solution to Problem 2.3(d).

(c) The formula that results from substituting the term $(x \cdot t)$ for the free occurrences of x is

$$(\forall x\, \exists t\, x' = t' \lor \exists y\,((x \cdot t) + y) = t).$$

The term may be freely substituted for x.

3

1	(1)	$\forall x\, \forall y\, (x + y) = (y + x)$	Ass
1	(2)	$\forall y\, (x + y) = (y + x)$	UE, 1
1	(3)	$(x + \mathbf{0}) = (\mathbf{0} + x)$	UE, 2
1	(4)	$\forall x\, (x + \mathbf{0}) = (\mathbf{0} + x)$	UI, 3

Note that we cannot use the UE Rule to go directly from line 1 to line 4, since the UE Rule enables us only to eliminate a universal quantifier that occurs at the beginning of a formula. So we cannot immediately eliminate the quantifier $\forall y$ from the formula on line 1.

4

1	(1)	$\forall v\,(\phi \rightarrow \psi)$	Ass
1	(2)	$(\phi \rightarrow \psi)$	UE, 1
3	(3)	ϕ	Ass
1, 3	(4)	ψ	Taut, 2, 3
1, 3	(5)	$\forall v\,\psi$	UI, 4
1	(6)	$(\phi \rightarrow \forall v\,\psi)$	CP, 5

Thus

$$\forall v\,(\phi \rightarrow \psi) \vdash (\phi \rightarrow \forall v\,\psi)$$

provided the variable v does not occur freely in ϕ.

The condition that v does not occur freely in ϕ is needed to derive line 5, because the application of the UI Rule to the formula on line 4 requires that v does not occur freely in any of the assumptions in force on line 4, and ϕ is one of these.

5

1	(1)	$\forall v\,(\phi \rightarrow \psi)$	Ass
2	(2)	$\forall v\,(\psi \rightarrow \theta)$	Ass
1	(3)	$(\phi \rightarrow \psi)$	UE, 1
2	(4)	$(\psi \rightarrow \theta)$	UE, 2
1, 2	(5)	$(\phi \rightarrow \theta)$	Taut, 3, 4
1, 2	(6)	$\forall v\,(\phi \rightarrow \theta)$	UI, 5

Section 3

1

1	(1)	$\forall v\,(\phi \rightarrow \psi)$	Ass
2	(2)	$\exists v\,\phi$	Ass
3	(3)	ϕ	Ass
1	(4)	$(\phi \rightarrow \psi)$	UE, 1
1, 3	(5)	ψ	Taut, 3, 4
1, 2	(6)	ψ	EH, 5
1	(7)	$(\exists v\,\phi \rightarrow \psi)$	CP, 6

The condition that v does not occur freely in ψ (the formula on line 5) is needed to justify the use of the EH Rule to obtain line 6.

2

1	(1)	$\forall v\,(\theta \rightarrow \psi)$	Ass
1	(2)	$(\theta \rightarrow \psi)$	UE, 1
3	(3)	$\exists v\,\theta$	Ass
1	(4)	$(\theta \rightarrow (\theta \rightarrow \psi))$	Taut, 2
5	(5)	θ	Ass
1, 5	(6)	$(\theta \rightarrow \psi)$	Taut, 4, 5
1, 5	(7)	$\exists v\,(\theta \rightarrow \psi)$	EI, 6
1, 3	(8)	$\exists v\,(\theta \rightarrow \psi)$	EH, 7
1	(9)	$(\exists v\,\theta \rightarrow \exists v\,(\theta \rightarrow \psi))$	CP, 8

3 The mistake is in the use of the EH Rule on line 7. In general, the variable v will have free occurrences in the formula ψ. This formula is used as an additional assumption on line 6. So the use of the EH Rule to derive line 7 is not legitimate. (On the other hand, there is nothing wrong with the use of the EH Rule on line 8, since in this case the additional assumption on which line 7 depends, that is, the formula $\exists v\,\phi$, does not contain any free occurrences of the variable v.)

4

1	(1)	$\exists x\, x = \mathbf{0}$	Ass
2	(2)	$x = \mathbf{0}$	Ass
2	(3)	$\exists y\, y = \mathbf{0}$	EI, 2
1	(4)	$\exists y\, y = \mathbf{0}$	EH, 3

5 Let ψ and ϕ both be the formula $v = \mathbf{0}'$, which contains a free occurrence of v. Then trivially ψ is a logical consequence of ϕ, as they are the same formula. But ψ is not a logical consequence of $\exists v\, \phi$. For instance, take the standard interpretation $\mathscr{N}$ and give any free occurrences of v the value 3. Then $\exists v\, v = \mathbf{0}'$ is true (as the occurrence of v in this formula is not free, and the formula is true as there is indeed a number in $\mathbb{N}$ equal to the successor of 0) while $v = \mathbf{0}'$ is false, because v is being interpreted by 3 rather than 1.

6 Let ϕ be the formula $\forall y\, v = y$ and let τ be the term y, which is not freely substitutable for v in ϕ. The formula $\phi(\tau/v)$ is then $\forall y\, y = y$, which is true in all interpretations. But $\exists v\, \phi$ is the formula $\exists v\, \forall y\, v = y$ which is false in the standard interpretation $\mathscr{N}$, or indeed in any interpretation with a domain consisting of more than one element.

INDEX

Ass 9
CP 11
EH 31
EI 29
Taut 10
UE 23
UI 24

$\phi(\tau/v)$ 20
$\phi_1, \phi_2, \ldots, \phi_k \vdash \psi$ 26
$\vdash$ 26

assertion 8
Assumption Rule 9
assumptions in force 7

bound occurrence 17
bound variable 17

Conditional Proof Rule 11

depends 7
derivation 4
derive 4
dummy variable 17, 19

existential formula 28
Existential Hypothesis Rule 31
Existential Quantifier Introduction Rule 29

formal proof 8
free occurrence 17
free substitution 22
free variable 17

justification 8

logical validity 9

machine checkability 9

quantifier logic 16

rule of proof 8

schematic proof 13
sentence 19

Tautology Rule 10
turnstile symbol 26

universal formula 24
Universal Quantifier Elimination Rule 23
Universal Quantifier Introduction Rule 24
universal statement 24